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Continuous Nowhere Differentiable Functions: Generalizations of Proofs

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Continuous Nowhere Differentiable Functions:

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Chapter 1

Introduction and Background Information

The task of finding functions which are continuous but nowhere differentiable has mystified and challenged mathematicians for the past three centuries. Bernard Bolzano is believed to have constructed the first example of a continuous nowhere differentiable function on an interval in 1830. Since then, several other mathematicians have constructed continuous functions which are nowhere differentiable on the entire set of real numbers or on a dense subset of the real numbers. In this paper, I will examine the work of Hermann Schwarz, Isaac Schoenberg, and Walter Rudin in this field. I will present and explain their original constructions and proofs of continuous functions which are nowhere differentiable or non-differentiable on a dense subset of their domains, and then present a generalization of their functions and proofs. Throughout this paper, we utilize the following notation:

$\mathcal{N}$ represents the Natural Numbers
$\mathcal{R}$ represents the Real Numbers

In order to construct these functions and prove their continuity and non-differentiability, we need a working definition of pointwise convergence and uniform convergence for a sequence of functions, and uniform convergence of a series of functions. We also need to utilize several theorems about the convergence of sequences/series of functions, namely:

1. The Cauchy criterion for uniform convergence of a sequence of functions
2. Theorem: If $\{f_n\}$ is a sequence of continuous functions on E, and f_n converges uniformly to f on E, then f is a continuous function on E.
3. Theorem: The Weierstrass M-Test
4. Lemma: Let $a < a_n < x < b_n < b$ for all $n \in \mathcal{N}$, and let $a_n \to x$ and $b_n \to$ x. If f:$[a, b] \to \mathcal{R}$ is a continuous function and $f'(x)$ exists, then
$$\lim_{n\to\infty} \frac{f(b_n)-f(a_n)}{b_n-a_n} = f'(x)$$

In the following pages, I will provide the definitions and examples of pointwise convergence and uniform convergence for a sequence of functions, the definition of uniform convergence of a series of functions, and proofs of the above 4 theorems and lemmas. The sources I used for these definitions and proofs are:

- Rudin, W. (1976). *Principles of Mathematical Analysis, Third Edition*. New York, McGraw-Hill, Inc.

- Thim, J. (2003). *Continuous Nowhere Differentiable Functions* (2003:320 CIV). [Master's Thesis, Lulea University of Technology].
- Whitaker, John. Shawnee State University, 2022, Mathematical Analysis II, You Tube, https://www.youtube.com/watch?v=s2c44HEPiTc&t=9s.

I. Pointwise convergent sequence of functions

Definition:

Suppose $\{f_n\}$ is a sequence of functions defined on a set E. And suppose that the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges for any x in E. Then we define

$$f(x) = lim_{n\to\infty} f_n(x)$$

and we say that $\{f_n\}$ converges pointwise to the function f on E.

In other words, a sequence of functions converges to the function f pointwise on E if for every x in E and for any $\varepsilon > 0$, there is an N $\in\mathcal{N}$ such that for any $n \geq$ N,

$$|f_n(x) - f(x)| \leq \varepsilon$$

In pointwise convergence, N depends on one's choice of ε and x.

Example:

The sequence $\{f_n(x)\} = \{x^n\}$ for $x \in (0,1)$ is a geometric sequence with $|x| < 1$. Therefore, it converges to 0.

For example, choose $\varepsilon = \frac{1}{10}$ and x_o=$\frac{1}{2}$. Then for $n \geq 4$, $\left|\left(\frac{1}{2}\right)^n - 0\right| \leq \frac{1}{10}$.

II. Uniformly convergent sequence of functions

Definition: We say that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly on E to a function $\boldsymbol{f}$ if for each $\varepsilon > 0$, there is an integer N such that for any $n \geq N$, $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in E$. *If a sequence converges uniformly, N does not depend on the value of x.*

Example:

$f_n(x) = x^n$ converges uniformly to 0 on the interval $\left[0, \frac{1}{2}\right]$

Proof:

Let $\varepsilon > 0$ *be given*. We need N∈ $\mathcal{N}$ such that for any n≥ N, $|f_n(x) - 0| \leq \varepsilon$, for *all* $x \in \left[0, \frac{1}{2}\right]$

So we need $|x^n - 0| \leq \varepsilon$. So we need $x^n \leq \varepsilon$.

Now, since $x \in [0, \frac{1}{2}]$, then $x^n \leq \left(\frac{1}{2}\right)^n$. So we want $\left(\frac{1}{2}\right)^n \leq \varepsilon$.

Solving this inequality using natural logs, we get:

$$n \geq \frac{ln(\varepsilon)}{ln\left(\frac{1}{2}\right)}$$

Since $\mathcal{N}$ is not bounded above, there is an N∈$\mathcal{N}$ such that $N \geq \frac{ln(\varepsilon)}{ln(\frac{1}{2})}$

So for any n≥ N, $|x^n - 0| \leq \varepsilon$, *for all* $x \in \left[0, \frac{1}{2}\right]$.

III. Theorem: The Cauchy Criterion for Uniform Convergence

The sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ defined on E converges uniformly on E if and only if for any $\varepsilon > 0$, there is an $N \in \mathcal{N}$ such that for any $n, m \geq N$,

$|f_n(x) - f_m(x)| < \varepsilon$ for all $x \in E$.

Proof→:

Suppose $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly on E to f(x). Let $\varepsilon > 0$ *be given*.

Then there is an N∈$\mathcal{N}$ *such that* $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$, for any n≥ N and all $x \in E$.

Thus, for any $n, m \geq N$ *and for all* $x \in E$, and applying the triangle inequality,

$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Proof←:

Suppose that for any ε > 0, there is an N $\in \mathcal{N}$ such that for any $n, m \geq N$,

$|f_n(x) - f_m(x)| \leq \varepsilon$ for all $x \in E$.

Then for each $x \in E$, $\{f_n(x)\}_{n=1}^{\infty}$ forms a Cauchy sequence which converges in $\mathcal{R}$.

Define $f(x) = \lim_{n\to\infty} f_n(x)$. We want to show that $\{f_n\}$ converges uniformly to f.

Let $\varepsilon > 0$ *be given*.

To show uniform convergence, we need an N∈ $\mathcal{N}$ such that for any n≥ N and for all $x \in E$,

$$|f_n(x) - f(x)| \leq \varepsilon.$$

By the assumption for $\leftarrow$, we know that there is an $\mathrm{N} \in \mathcal{N}$ such that for any $n, m \geq N$, then $|f_n(x) - f_m(x)| \leq \varepsilon$ for all $x \in E$.

Fix $n \geq N$. *Consider* $\lim_{m\to\infty} |f_n(x) - f_m(x)|$

Claim: $\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|$ *for all* $x \in E$.

This is equivalent to saying:

For a given $\delta > 0$, *there is a* $K \in \mathcal{N}$ *such that for any* $m \geq K$,

$$\big||f_n(x) - f_m(x)| - |f_n(x) - f(x)|\big| \leq \delta \text{ for all } x \in E.$$

Proof of claim:

Let $\delta > 0$ *be given. We need* $K \in \mathcal{N}$ *such that for any* $m \geq K$,

$$\big||f_n(x) - f_m(x)| - |f_n(x) - f(x)|\big| \leq \delta \text{ for all } x \in E.$$

It can be shown:

$$\big||f_n(x) - f_m(x)| - |f_n(x) - f(x)|\big| \leq |f_n(x) - f_m(x) - (f_n(x) - f(x))| = |f_m(x) - f(x)|$$

Since $\{f_m(x)\}$ *is Cauchy,* $\{f_m(x)\}$ *converges to* $f(x)$ *for each* $x \in E$.

So there is a $K_x \in \mathcal{N}$ *such that for any* $m \geq K_x$, $|f_m(x) - f(x)| \leq \delta$

Thus, we have proven the claim that $\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \leq \delta$ *for* $x \in E$.

Now since $|f_n(x) - f_m(x)| \leq \varepsilon$ for any $n, m \geq N$ *and for all* $x \in E$,

$$\text{then } \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

Note that for all x, that limit is the same, i.e., $\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|$ as shown in the proof of the claim.

Thus $|f_n(x) - f(x)| \leq \varepsilon$ for any $n \geq N$ and for all $x \in E$.

Thus $\{f_n(x)\}$ *converges uniformly on* E.

IV. Theorem: If $\{f_n\}$ is a sequence of continuous functions on E, and f_n converges uniformly to f on E, then f is a continuous function on E.

Proof:

Let $x_o \in E$.

Let $\{f_n\}$ be a sequence of continuous functions on E which converge uniformly to f on E.

Because f_n converges uniformly, we can say:

For any $\varepsilon > 0$, there is an N $\in \mathcal{N}$ such that for any $n \geq$ N,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \text{ for all } x \in E.$$

Because f_n is continuous, we can say:

For any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|x - x_o| < \delta \quad \text{implies that } |f_n(x) - f_n(x_o)| < \frac{\varepsilon}{3}$$

Let $\varepsilon > 0$ be given. Let $x \in E$. Let n$\in \mathcal{N}$ with $n \geq N$. Let $|x - x_o| < \delta$. Then

$$\begin{aligned}|f(x) - f(x_o)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_o) + f_n(x_o) - f(x_o)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_o)| + |f_n(x_o) - f(x_o)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

Since $|f(x) - f(x_o)| \leq \varepsilon$ whenever $|x - x_o| < \delta$, then f is continuous at x_o.

Since our choice of x_o was arbitrary, f is continuous on E.

V. Uniformly Convergent Series of Functions

Before we give the statement and proof of the Weierstrass-M Test, we give a preparatory definition of *Uniformly Convergent Series of Functions*:

Definition:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on E.

We say that $\sum_{n=1}^{\infty} f_n$ converges uniformly on E iff

$\{\sum_{i=1}^{n} f_i\}_{n=1}^{\infty}$ converges uniformly on E.

VI. Theorem: Weierstrass-M Test

Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \leq M_n \quad (x \in E,\ n \in \mathcal{N}).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof:

Let $\{f_n\}$ be a sequence of functions defined on E.

Suppose $|f_n(x)| \leq M_n \quad (x \in E,\ n \in \mathcal{N})$, and suppose $\sum M_n$ converges.

Let $\varepsilon > 0$ be given.

Since $\sum M_n$ converges, there is an N∈$\mathcal{N}$ such that for $m \geq n \geq N$, $\sum_{i=n}^{m} M_i \leq \varepsilon$
Cauchy Criterion for convergent series

Since $|f_n(x)| \leq M_n$, then $|\sum_{i=n}^{m} f_i(x)| \leq \sum_{i=n}^{m}|f_i(x)| \leq \sum_{i=n}^{m} M_i$

Let $S_n(x) = \sum_{i=1}^{n} f_i(x)$

For any $m \geq n \geq N$, $|S_m(x) - S_n(x)| = |\sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{n} f_i(x)|$

$$= |\sum_{i=n+1}^{m} f_i(x)|$$
$$\leq \sum_{i=n+1}^{m} |f_i(x)|$$
$$\leq \sum_{i=n}^{m} |f_i(x)|$$
$$\leq \sum_{i=n}^{m} M_i \leq \varepsilon$$

So for any $m \geq n \geq N$, and for any $x \in E$, $|S_m(x) - S_n(x)| < \varepsilon$

So by the Cauchy Criterion for Uniform Convergence,

$\{S_n\}_{n=1}^{\infty} = \left\{ \sum_{i=1}^{n} f_i(x) \right\}_{n=1}^{\infty}$ converges uniformly.

Therefore, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly. (See definition above)

VII. Lemma: ***Let $a < a_n < x < b_n < b$ for all n ∈$\mathcal{N}$, and let $a_n \to x$ and $b_n \to x$.***

If f:$[a, b] \to \mathcal{R}$ is a continuous function and $f'(x)$ exists, then

$$\lim_{n\to\infty} \frac{f(b_n)-f(a_n)}{b_n-a_n} = f'(x)$$

Proof:

Note that $\left|\frac{b_n-x}{b_n-a_n}\right| < \frac{b_n-a_n}{b_n-a_n} = 1$ and $\left|\frac{x-a_n}{b_n-a_n}\right| < \frac{b_n-a_n}{b_n-a_n} = 1$

Now we can estimate $\left|\frac{f(b_n)-f(a_n)}{b_n-a_n} - f'(x)\right|$ by expressing it as the sum of two differences, and multiplying each difference by the fraction of the total interval, $|b_n - a_n|$, which it represents. So,

$\left|\frac{f(b_n)-f(a_n)}{b_n-a_n} - f'(x)\right| =$

$$= \left| \frac{b_n - x}{b_n - a_n}\left(\frac{f(b_n) - f(x)}{b_n - x} - f'(x)\right) + \frac{x - a_n}{b_n - a_n}\left(\frac{f(a_n) - f(x)}{a_n - x} - f'(x)\right) \right|$$

$= \left|\frac{b_n-x}{b_n-a_n}\right| \cdot \left|\frac{f(b_n)-f(x)}{b_n-x} - f'(x)\right| + \left|\frac{x-a_n}{b_n-a_n}\right| \cdot \left|\frac{f(a_n)-f(x)}{a_n-x} - f'(x)\right|$ since all factors are positive

$\leq \left|\frac{f(b_n)-f(x)}{b_n-x} - f'(x)\right| + \left|\frac{f(a_n)-f(x)}{a_n-x} - f'(x)\right|$ since $\left|\frac{b_n-x}{b_n-a_n}\right| \leq 1$ and $\left|\frac{x-a_n}{b_n-a_n}\right| \leq 1$

Note: $\lim_{n\to\infty} \left|\frac{f(b_n)-f(x)}{b_n-x} - f'(x)\right| = 0$

and $\lim_{n\to\infty} \left|\frac{f(a_n)-f(x)}{a_n-x} - f'(x)\right| = 0$ by the Sequential Criterion for Function Limits

So $\lim_{n\to\infty} \left|\frac{f(b_n)-f(x)}{b_n-x} - f'(x)\right| + \left|\frac{f(a_n)-f(x)}{a_n-x} - f'(x)\right| = 0 + 0 = 0$

Therefore, $\lim_{n\to\infty} \left|\frac{f(b_n)-f(a_n)}{b_n-a_n} - f'(x)\right| = 0$ Substituting in a positive lesser value

Therefore, $\lim_{n\to\infty} \frac{f(b_n)-f(a_n)}{b_n-a_n} = f'(x)$

Chapter 2

Schwarz' Function: Original Proof and Generalization

Hermann Schwarz, a German mathematician and student of Karl Weierstrass, published his proof of a continuous function which is non-differentiable on a dense subset of the real numbers in 1873. The proof is given below.

Theorem: *The Schwarz function* $S: (0, M) \to \mathscr{R}$ *defined by*
$S(x) = \sum_{k=0}^{\infty} \frac{\varphi(2^k x)}{4^k}$, *where* $\varphi(x) = [x] + \sqrt{x - [x]}$, $[x]$ *= the greatest integer* $\leq x$,
is continuous and non-differentiable on a dense subset of $(0, M)$, *with* $M > 0$ *and* $M \in \mathscr{R}$*.*

Proof:

I. Let $\boldsymbol{S_k(x) = \frac{\varphi(2^k x)}{4^k}}$**. Prove that** $\{\boldsymbol{S_k(x)}\}$ **is a sequence of continuous functions on** $(\boldsymbol{0}, \boldsymbol{M})$:

First we examine the continuity of $\varphi(x) = [x] + \sqrt{x - [x]}$.

This is the sum of the greatest integer function and the composition of the square root function with the function $f(x) = x - [x]$.

The square root function and $y = x$ are continuous for all values of $x \in (0, M)$.

Also, the greatest integer function is continuous for all x, except possibly when $x \in \mathcal{N}$.

So $\varphi(x)$ will be continuous for all x, except possibly when $x \in \mathcal{N}$, as it is the sum and composition of continuous functions.

So the only possible discontinuities for the function $\varphi(x)$ might occur when $x \in \mathcal{N}$. So we *will investigate this case:*

Let $p \in \mathcal{N}$*. We need to investigate right and left limits of* $\varphi(x)$ *as* $x \to p$.

$$\lim_{x \to p^+} \varphi(x) = \lim_{x \to p^+} \left([x] + \sqrt{x - [x]}\right) = p + \sqrt{p - p} = p$$

$$\lim_{x \to p^-} \varphi(x) = \lim_{x \to p^-} \left([x] + \sqrt{x - [x]}\right) = (p - 1) + \sqrt{p - (p - 1)} = (p - 1) + \sqrt{1} = p$$

Since $\lim_{x \to p^+} \varphi(x) = \lim_{x \to p^-} \varphi(x) = p$, *then* $\varphi(x)$ *is continuous on* $(0, \infty)$.

Now $\varphi(2^k x)$ *is the composition of continuous functions, so it is continuous.*

So $S_k(x) = \frac{\varphi(2^k x)}{4^k}$ *is the quotient of continuous functions, so it is continuous.*

Therefore, $\{S_k(x)\}$ *is a sequence of continuous functions on* $(0, \infty)$.

Therefore, $\{S_k(x)\}$ is a sequence of continuous functions on $(0, M)$.

Furthermore, $\varphi(p) = p$ for all $p \in \mathcal{N}$.

II. Prove that the series converges uniformly.

Let $x \in \mathcal{R},\ x \in (0, M)$.

Then x can be written as $x = p + h$, where $h \in \mathcal{R},\ h \in (0,1)$ and $p \in \{\mathcal{N} \cup \{0\}\}$

Then $\varphi(p+h) = [p+h] + \sqrt{(p+h) - [p+h]}$

$$= p + \sqrt{(p+h) - p}$$

$$= p + \sqrt{h}$$

Now define $q(h) = \varphi(p+h) - (p+h)$

So $q(h) = p + \sqrt{h} - p - h = \sqrt{h} - h = (h)^{\frac{1}{2}} - h$

So $q'(h) = \frac{1}{2}(h)^{\frac{-1}{2}} - 1 = \frac{1}{2\sqrt{h}} - 1$

We set $q'(h) = 0$ to find critical points of $q(h)$:

Solving $\frac{1}{2\sqrt{h}} - 1 = 0$, we get $h = \frac{1}{4}$.

Next we find $q"(h) = -\frac{1}{4}(h)^{\frac{-3}{2}}$.

Now $q"\left(\frac{1}{4}\right) < 0$. So when $h = \frac{1}{4}$, $q(h)$ is concave down. Therefore, $q(h)$ reaches its maximum value at $h = \frac{1}{4}$.

So $q(h) \leq q\left(\frac{1}{4}\right)$. Note that $q\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} - \frac{1}{4} = \frac{1}{4}$

So $\varphi(p+h) - (p+h) \leq q\left(\frac{1}{4}\right)$ Substitute $\varphi(p+h) - (p+h)$ for $q(h)$ as defined above

So $\varphi(p+h) \leq (p+h) + q\left(\frac{1}{4}\right)$.

So $\varphi(p+h) \leq (p+h) + \frac{1}{4}$ Substitute in $\frac{1}{4}$ for $q\left(\frac{1}{4}\right)$ as derived above

So $\varphi(x) \leq x + \frac{1}{4}$ Substitute in x for $p + h$ as defined above

Now consider $\left|\frac{\varphi(2^k x)}{4^k}\right|$:

$$\left|\frac{\varphi(2^k x)}{4^k}\right| = \frac{\varphi(2^k x)}{4^k} \quad \text{since } \varphi > 0, x > 0$$

$$\leq \frac{2^k x + \frac{1}{4}}{4^k} \qquad \text{since } \varphi(x) \leq x + \tfrac{1}{4} \text{ (from above)}$$

$$= \frac{x}{2^k} + \frac{1}{4^{k+1}}$$

$$< \frac{M}{2^k} + \frac{1}{4^{k+1}} \qquad \text{since } x < M$$

So for all n, $\left|\frac{\varphi(2^k x)}{4^k}\right| < \frac{M}{2^k} + \frac{1}{4^{k+1}}$

So each term in the sequence $\{S_k(x)\}$ is bounded above by $\frac{M}{2^k} + \frac{1}{4^{k+1}}$

Now $\sum_{k=0}^{\infty} \frac{M}{2^k} + \frac{1}{4^{k+1}}$ is the sum of 2 geometric series with $r < 1$, and so it converges.

Then by the Weierstrass-M Test, $\sum_{n=0}^{\infty} \frac{\varphi(2^k x)}{4^k}$ converges uniformly to $S(x)$.

Define $\{f_n\} = \left\{\sum_{k=1}^{n} \frac{\varphi(2^k x)}{4^k}\right\}_{n=1}^{\infty}$ to be the sequence of partial sums of $S(x)$.

Then $\{f_n\}$ converges uniformly on $(0, M)$ to $S(x)$, since the corresponding series converges uniformly to $S(x)$.
Also, each $f_n(x)$ is the sum of continuous functions, so it is continuous.

Therefore, since $\{f_n(x)\}$ is a sequence of continuous functions on $(0, M)$ which converges uniformly to $S(x)$, we can conclude that $S(x)$ is continuous on $(0, M)$.
(See theorem IV in Background Information)

III. Show that S is not differentiable on a dense subset of $(0, M)$:

Let x_o *and* $x_1 \in (0, M)$. *Without loss of generality, let* $x_o < x_1$.

We want to show that there is an x $(x_o < x < x_1)$, such that $S'(x)$ does not exist.

Proof:

Let x be a dyadic rational number such that $x_o < x < x_1$.

Let $x = \frac{j}{2^m} = j \times 2^{-m}$ for some $j, m \in \mathcal{N}$. Let $0 < h < \frac{1}{2^m}$

Consider $\frac{S(x+h) - S(x)}{h} = \sum_{n=0}^{\infty} \frac{\varphi(2^n(x+h)) - \varphi(2^n x)}{4^n h}$

$$\geq \frac{\varphi(2^m(x+h)) - \varphi(2^m x)}{4^m h}$$ *since each term in the series is non-negative, the series ≥ one term*

From how we defined $\boldsymbol{j}$ *and* $\boldsymbol{h}$ *above, we can derive:*

$2^m x = j\ for\ j \in \mathcal{N}$ *and* $2^m h < 1$

Therefore, $[2^m x + 2^m h] = [2^m x]$=*j*

So $\varphi\big(2^m(x+h)\big) - \varphi(2^m x) =$

$$= \varphi\,(2^m x + 2^m h) - \varphi(2^m x)$$

$$= [2^m x + 2^m h] + \sqrt{(2^m x + 2^m h) - [2^m x + 2^m h]} - [2^m x] - \sqrt{2^m x - [2^m x]}$$

$$= j + \sqrt{j + 2^m h - j}\ - j\ - \sqrt{j - j} = \sqrt{2^m h}$$

Now $\frac{S(x+h)-S(x)}{h} \geq \frac{\varphi\left(2^m(x+h)\right)-\varphi(2^m x)}{4^m h}$

$$= \frac{\sqrt{2^m h}}{4^m h} \ = \frac{1}{2^m\sqrt{2^m}} \times \frac{1}{\sqrt{h}}$$

So $\lim\limits_{h\to 0} \frac{S(x+h)-S(x)}{h} \ \geq \ \lim\limits_{h\to 0} \left(\frac{1}{2^m\sqrt{2^m}} \times \frac{1}{\sqrt{h}}\right) = \infty$

Therefore, $S'(x) does\ not\ exist, and\ \ S$ is non-differentiable on a dense subset of *(0,M)*.

Note: The above proof was adapted from the proof found in the following document: Thim, J. (2003). *Continuous Nowhere Differentiable Functions* (2003:320 CIV). [Master's Thesis, Lulea University of Technology]. I have expanded the proof to provide more in-depth proofs of several steps.

Generalized Schwarz' Proof

I have generalized the Schwarz function and have proven that the generalized function is continuous and non-differentiable on a dense subset of the real numbers, as explained below.

Theorem (generalized): *The generalized Schwarz function* $S\colon (0, M) \to \mathcal{R}$ *defined by*

$S(x) = \sum_{k=0}^{\infty} \frac{\varphi(a^k x)}{b^k}$, *where* $\varphi(x) = [x] + \sqrt[n]{x - [x]}$, $[x]$ *is the greatest integer* $\leq x$,

$a \in \mathcal{R},\ b \in \mathcal{R},\ b > a > 1\ ,\ n \in \mathcal{N}\ ,\ n \geq 2$

is continuous and non-differentiable on a dense subset of $(0, M)$, *with* $M > 0$ *and* $M \in \mathcal{R}$.

Changes from original proof:

- In definition of $S(x)$, change $\frac{\varphi(2^k x)}{4^k}$ to $\frac{\varphi(a^k x)}{b^k}$; $a \in \mathcal{R};\ b \in \mathcal{R};\ b > a > 1$
- In definition of $\varphi(x)$, change $[x] + \sqrt{x - [x]}$ to $[x] + \sqrt[n]{x - [x]}$; $n \in \mathcal{N};\ n \geq 2$
- In part III of proof, change $x = \frac{\boldsymbol{j}}{\boldsymbol{2^m}}$ to $x = \frac{\boldsymbol{j}}{\boldsymbol{a^m}}$ *and change 0<* $h < \frac{1}{2^m}$ *to 0<* $h < \frac{1}{a^m}$.

Proof:

I. Let $S_k(x) = \frac{\varphi(a^k x)}{b^k}$. Prove that $\{S_k(x)\}$ is a sequence of continuous functions on $(0, M)$:

Since $\frac{1}{b^k}$ is continuous, we need only to examine the numerator*:*

$$\varphi(a^k x) = [a^k x] + \sqrt[n]{a^k x - [a^k x]}$$

Consider $\varphi(t) = [t] + \sqrt[n]{t - [t]}\,;\qquad t \in \mathcal{R}$

Now since the greatest integer function is continuous for all t, except possibly when $t \in \mathcal{N}$, $\varphi(t)$ will be continuous for all t, except possibly when $t \in \mathcal{N}$.

So the only possible discontinuities for the function $\varphi(t)$ might occur when $t \in \mathcal{N}$. So we will investigate this case:

Let $p \in \mathcal{N}$. We need to investigate right and left limits of $\varphi(t)$ as $t \to p$.

$$\lim_{t \to p^+} \varphi(t) = \lim_{t \to p^+} \left([t] + \sqrt[n]{t - [t]}\right) = p + \sqrt[n]{p - p} = p$$

$$\lim_{t \to p^-} \varphi(t) = \lim_{t \to p^-} \left([t] + \sqrt[n]{t - [t]}\right) = (p-1) + \sqrt[n]{p - (p-1)} = (p-1) + \sqrt[n]{1} = p$$

Since $\lim_{t \to p^+} \varphi(t) = \lim_{t \to p^-} \varphi(t) = p$, then $\varphi(t)$ is continuous on $(0, \infty)$ for all t.

Now $\boldsymbol{\varphi(a^k x)}$ is the sum and composition of continuous functions, so it is continuous on $(0, \infty)$

Also $\boldsymbol{S_k(x)} = \frac{\boldsymbol{\varphi(a^k x)}}{\boldsymbol{b^k}}$ is the quotient of continuous functions, so it is continuous on $(0,\infty)$

Therefore, $\{S_k(x)\}$ is a sequence of continuous functions on $(0, \infty)$.

Therefore, $\{S_k(x)\}$ is a sequence of continuous functions on $(0, M)$.

Furthermore, $\varphi(p) = p$ for all $p \in \mathcal{N}$.

II. Prove that the series converges uniformly.

Let $x \in \mathcal{R},\ x \in (0, M)$.

Then x can be written as $x = p + h$, where $h \in \mathcal{R},\ h \in (0,1)$ and $p \in \{\mathcal{N} \cup \{0\}\}$

Then $\varphi(p + h) = [p + h] + \sqrt[n]{(p + h) - [p + h]}$

$$= p + \sqrt[n]{(p + h) - p}$$

$$= p + \sqrt[n]{h}$$

Now define $q(h) = \varphi(p + h) - (p + h)$

So $q(h) = p + \sqrt[n]{h} - p - h = \sqrt[n]{h} - h = (h)^{\frac{1}{n}} - h$

So $q'(h) = \frac{1}{n}(h)^{\left(\frac{1}{n}-1\right)} - 1$

We set $q'(h) = 0$ to find critical points of $q(h)$:

Solving $\frac{1}{n}(h)^{\left(\frac{1}{n}-1\right)} - 1 = 0$:

$\frac{1}{n}(h)^{\left(\frac{1}{n}-1\right)} = 1$

$(h)^{\left(\frac{1}{n}-1\right)} = n$

$ln(h)^{\left(\frac{1}{n}-1\right)} = ln(n)$

$\left(\frac{1}{n} - 1\right) ln\ (h) = ln(n)$

$ln(h) = \frac{ln(n)}{\frac{1}{n}-1} = \frac{n \cdot ln(n)}{1-n}$

So $h = e^{\left(\frac{n \cdot ln(n)}{1-n}\right)} = e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$ is a critical point

Next we find $q''(h) = \left(\frac{1}{n} - 1\right) \cdot \frac{1}{n} \cdot h^{\left(\frac{1}{n}-2\right)} = \left(\frac{1-n}{n^2}\right)\left(h^{\left(\frac{1}{n}-2\right)}\right)$

So $q''(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}) = \left(\frac{1-n}{n^2}\right)\left(\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)^{\left(\frac{1}{n}-2\right)}\right)$ This is negative since $\left(\frac{1-n}{n^2}\right) < 0$ and $e^x > 0$ for all x.

So when $h = e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$, $q(h)$ is concave down. Therefore, $q(h)$ reaches its maximum value at $h = e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$

.

So $q(h) \leq q\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$

Now $q(h) = \varphi(p+h) - (p+h)$ from above

So $\varphi(p+h) - (p+h) \leq q\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$ Substitution

So $\varphi(p+h) \leq (p+h) + q\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$

Now we evaluate $q\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$:

$q\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right) = \left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)^{\frac{1}{n}} - \left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$ from our definition of $q(h)$ above

$= e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$

Recall, $\varphi(p+h) \leq (p+h) + q\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$ from above

So $\varphi(p+h) \leq (p+h) + e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$ Substitute in for $q\left(e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$

Recall that $x = p + h$

So $\varphi(x) \leq x + e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$ Note that $e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)} > 0, since\ \ e^{\left(\frac{-ln(n)}{n-1}\right)} > e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$

Also, $e^{\left(\frac{-ln(n)}{n-1}\right)} < 1$ for $n > 1$. So $0 < e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)} < 1$

So $\varphi(a^k x) \leq (a^k x) + e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}$ Substitute $a^k x$ for x in above

So $\frac{\varphi(a^k x)}{b^k} \leq \frac{(a^k x) + e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}}{b^k}$ Since b^k is positive, we can divide each side by b^k

So $\left|\frac{\varphi(a^k x)}{b^k}\right| \leq \frac{(a^k x) + e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}}{b^k}$ Since $\frac{\varphi(a^k x)}{b^k} > 0$

$$= \left(\frac{a}{b}\right)^k x + \frac{e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}}{b^k}$$

$$< \left(\frac{a}{b}\right)^k M + \frac{e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}}{b^k}$$ since $x < M$

So for all k, $\left|\frac{\varphi(a^k x)}{b^k}\right| < \left(\frac{a}{b}\right)^k M + \frac{1}{b^k}\left(e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right) = M_k$

So each term of $\{S_k(x)\} = \left\{\frac{\varphi(a^k x)}{b^k}\right\}$ is bounded above by $M_k = \left(\frac{a}{b}\right)^k M + \frac{1}{b^k}\left(e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$.

Now consider $\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty}\left\{\left(\frac{a}{b}\right)^k M + \frac{1}{b^k}\left(e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)\right\}$

$$= \sum_{k=0}^{\infty}\left(\frac{a}{b}\right)^k M + \sum_{k=0}^{\infty}\frac{1}{b^k}\left(e^{\left(\frac{-ln(n)}{n-1}\right)} - e^{\left(\frac{-n \cdot ln(n)}{n-1}\right)}\right)$$

$$< \sum_{k=0}^{\infty}\left(\frac{a}{b}\right)^k M + \sum_{k=0}^{\infty}\frac{1}{b^k}(1)$$ See above for explanation of substitution made

This is the sum of 2 geometric series, each with $r < 1$, so it converges.

Therefore, $\sum_{k=0}^{\infty} M_k$ converges.

Therefore, $\sum_{k=0}^{\infty}\frac{\varphi(a^k x)}{b^k}$ converges uniformly on (0,M) to $S(x)$ by the Weierstrass-M Test.

Define $\{f_i\} = \left\{\sum_{k=1}^{i} \frac{\varphi(a^k x)}{b^k}\right\}_{i=1}^{\infty}$ to be the sequence of partial sums of $S(x)$.

Then $\{f_i\}$ converges uniformly on (0,M) to $S(x)$, since the corresponding series converges uniformly on (0,M) to $S(x)$.

Also, each $f_i(x)$ is the sum of continuous functions, it is also continuous.

Therefore, since $\{f_i(x)\}$ is a sequence of continuous functions on $(0, M)$ which converges uniformly to $S(x)$, we can conclude that $S(x)$ is continuous on $(0, M)$.

(See theorem IV in Background Information)

III. Show that S is not differentiable on a dense subset of $(0, M)$:

Let x_o and $x_1 \in (0, M)$. $Without\ loss\ of\ generality, let\ x_o < x_1$

We want to show that there is an x $(x_o < x < x_1)$, such that $S'(x)$ does not exist.

Proof:

Let $m \in \mathcal{N}$.

Then since $x_1 > x_{o,}$ $\quad a^m x_1 > a^m x_o$ Since *a is positive*

Since $a > 1$, there is a $m \in \mathcal{N}$, such that $a^m > \frac{1}{x_1 - x_o}$

So $a^m(x_1 - x_o) > 1$

So $a^m x_1 - a^m x_o > 1$

Therefore, there is some integer *j* such that $a^m x_o < j < a^m x_1$

Therefore, $\frac{a^m x_o}{a^m} < \frac{j}{a^m} < \frac{a^m x_1}{a^m}$ Dividing each term by a^m

Therefore $x_o < \frac{j}{a^m} < x_1$ Simplifying

Now fix $x = \frac{\boldsymbol{j}}{\boldsymbol{a^m}}$ such that $x_o < x < x_1$ and $j, m \in \mathcal{N}$. (We have just shown that such an x exists.)

So $x = j \cdot a^{-m}$. Fix h such that $0 < h < \frac{1}{a^m}$ *and* $h \in \mathfrak{R}$

Consider $\frac{S(x+h) - S(x)}{h} = \sum_{n=0}^{\infty} \frac{\varphi(a^n(x+h)) - \varphi(a^n x)}{b^n h}$

$$\geq \frac{\varphi(a^m(x+h)) - \varphi(a^m x)}{b^m h}$$ *since each term in the series is non-negative, the series $\geq$ one term*

*From how we defined **j** and **h** above, we can derive:*

$a^m x = j\ for\ j \in \mathcal{N}$ *and* $a^m h < 1$

Therefore, $[a^m x + a^m h] = [a^m x]$= *j*

So $\varphi\big(a^m(x+h)\big) - \varphi(a^m x) =$

$= \varphi\ (a^m x + a^m h) - \varphi(a^m x)$

$= [a^m x + a^m h] + \sqrt[n]{(a^m x + a^m h) - [a^m x + a^m h]} - [a^m x] - \sqrt[n]{a^m x - [a^m x]}$

$= j + \sqrt[n]{j + a^m h - j} - j\ - \sqrt[n]{j - j} = \sqrt[n]{a^m h}$

Now $\frac{S(x+h)-S(x)}{h} \geq \frac{\varphi(a^m(x+h))-\varphi(a^m x)}{b^m h}$ from a previous step

$= \frac{\sqrt[n]{a^m h}}{b^m h}$ Substitute in for $\varphi\ (a^m x + a^m h) - \varphi(a^m x)$

$= \frac{a^m h}{(b^m h)\left(\sqrt[n]{(a^m h)^{(n-1)}}\right)} = \frac{a^m}{(b^m)\left(\sqrt[n]{(a^m h)^{(n-1)}}\right)}$

$= \frac{a^m}{(b^m)\left(\sqrt[n]{(a^m)^{(n-1)}}\right)} \cdot \frac{1}{\sqrt[n]{h^{(n-1)}}}$

So $\lim\limits_{h\to 0} \frac{S(x+h)-S(x)}{h} \geq \lim\limits_{h\to 0} \left(\frac{a^m}{(b^m)\left(\sqrt[n]{(a^m)^{(n-1)}}\right)} \cdot \frac{1}{\left(\sqrt[n]{h^{n-1}}\right)}\right) = \infty$ *since* $\frac{a^m}{(b^m)\left(\sqrt[n]{(a^m)^{(n-1)}}\right)}$ *is fixed*

and as h$\to 0$, $\sqrt[n]{h^{n-1}} \to 0$, *so* $\frac{1}{\left(\sqrt[n]{h^{n-1}}\right)} \to \infty$

Therefore, $S'(x) does\ not\ exist, and\ \ S$ is non-differentiable on a dense subset of $(0, M)$.

Chapter 3

The Schoenberg Functions: Sagan's Proof and Generalization

In 1938, Isaac Schoenberg, a Romanian mathematician, extended the work done by Henri Lebesgue to construct two space-filling functions, $\varphi_s(x)$ and $\psi_s(x)$. (A space-filling function is a mapping of a line or 1-dimensional curve to every point in a 2-dimensional space.) Schoenberg proved that his functions are continuous. In 1992, Hans Sagan proved that these functions are also not differentiable on the closed interval [0,1], except possibly when $x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m}$ for some m; $a_i = \{0, 1, 2, \ldots 8\}$. Sagan's proof is given below:

Theorem: Schoenberg's two functions, $\varphi_s(x)$ and $\psi_s(x)$, defined below, are continuous and not differentiable on $[0, 1]$, except possibly when:

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

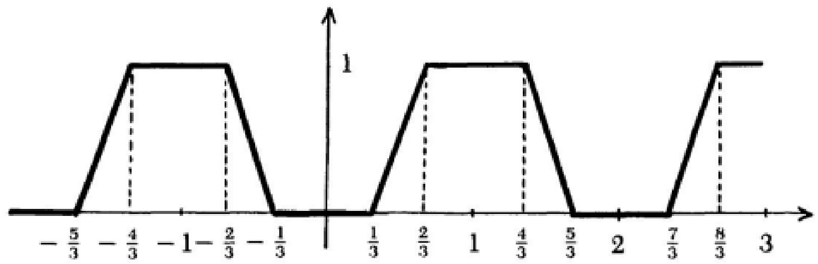
$$\varphi_s(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(3^{2k}x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(9^k x) \text{ and}$$

$$\psi_s(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(3^{2k+1}x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(3^{2k} \cdot 3x) = \varphi_s(3x)$$

where

$$\begin{array}{ll} p(x) = 0 & x \in \left[0, \frac{1}{3}\right] \\ p(x) = 3x - 1 & x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ p(x) = 1 & x \in \left[\frac{2}{3}, \frac{4}{3}\right] \\ p(x) = 5 - 3x & x \in \left[\frac{4}{3}, \frac{5}{3}\right] \\ p(x) = 0 & x \in \left[\frac{5}{3}, 2\right] \end{array}$$

and $p(x + 2) = p(x)$, for all $x \in \mathcal{R}$



Graph of Schoenberg's p function (from Hans Sagan, as cited below)

Note:

If l is an odd integer, and $x \in \left(l - \frac{1}{3}, l + \frac{1}{3}\right)$, *then* $p(x) = 1$

If l is an even integer, and $x \in \left(l - \frac{1}{3}, l + \frac{1}{3}\right)$, *then* $p(x) = 0$

Proof:

I. Show that $\varphi_s(x)$ is continuous on $[0,1]$

Let $\{s_k(x)\} = \left\{\frac{1}{2^k} p(9^k x)\right\}$ $\quad k \in \mathcal{N}$

$Then$ $\left|\frac{1}{2^k} p(9^k x)\right| \leq \frac{1}{2^k}$ for each k (since $0 \leq p(9^k x) \leq 1$)

So each term in the sequence is bounded above by $\frac{1}{2^k}$.

We know that $\sum_{k=0}^{\infty} \frac{1}{2^k}$ is an infinite geometric series with r< 1, and so it converges.

Since the series of upper bounds converges, then by the Weierstrass-M test,

$\sum_{k=0}^{\infty} \frac{1}{2^k}\ p(9^k x)$ converges uniformly on [0,1].

Therefore, $\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} p(9^k x)$ also converges uniformly on [0,1] to $\varphi_s(x)$.

Now define $\{f_n\} = \left\{\frac{1}{2} \sum_{i=1}^{n} s_i(x)\right\}_{n=1}^{\infty}$ to be the sequence of partial sums of $\varphi_s(x)$.

Then $\{f_n\}$ converges uniformly on [0,1] to $\varphi_s(x)$ since the corresponding series converges uniformly to $\varphi_s(x)$.

Note that each $s_k(x) = \frac{1}{2^k} p(9^k x)$ is the product and composition of continuous functions, and therefore it is continuous.

So each $f_n(x)$is the sum of continuous functions, so it is continuous.

Since $\{f_n\}$ is a sequence of continuous functions which converges uniformly on [0,1] to $\varphi_s(x)$, we can conclude that:

$\varphi_s(x) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^k x)$ is continuous on [0,1]. See IV in Background Information.

Similarly, $\psi_s(x)$ is also continuous on $[0,1]$.

II. Show that $\varphi_s(x)$ is not differentiable on (0,1), except possibly when:

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

(Proof by contradiction)

Let $t \in (0,1)$. Assume that $\varphi'_s(t)$ exists.

Then by a previous lemma (*VII in Background Information*),

if $0 < a_n < t < b_n < 1$, and $a_n \to t$ as $n \to \infty$, and $b_n \to t$ as $n \to \infty$, then

$$\lim_{n\to\infty}\frac{\varphi_s(b_n) - \varphi_s(a_n)}{b_n - a_n} = \varphi'_s(t)$$

We will construct 2 sequences, $\{a_n\}$ and $\{b_n\}$ which contradict our assumption:

Let $\widehat{k_n} = [9^n t]$ where $[x]$=the integer part of x, and t not of the form:

$$\frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Note: This restriction on the form of t is required so that $[9^n t]$ is always strictly less than $9^n t$.

Let $\widehat{a_n} = \widehat{k_n} \cdot 9^{-n} = [9^n t](9^{-n})$

Let $\widehat{b_n} = \widehat{k_n} \cdot 9^{-n} + 9^{-n} = [9^n t](9^{-n}) + 9^{-n}$

So there are either infinitely many $\widehat{k_n}$ which are odd or infinitely many $\widehat{k_n}$ which are even.

Case i: There are infinitely many even $\widehat{k_n}$

Let $\{k_n\}$ be a subsequence of $\{\widehat{k_n}\}$ such that k_n is even.

Let $\{a_n\}$ and $\{b_n\}$ be the corresponding sequences.

$\varphi_s(b_n) - \varphi_s(a_n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^k b_n) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^k a_n)$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\left(9^k(k_n 9^{-n} + 9^{-n})\right) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^k k_n 9^{-n})$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^{k-n}k_n + 9^{k-n}) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^{k-n}k_n)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot \left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right)$$ *Now, express this as 2 sums:*

$$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\left(p\left(9^{k-n}k_n + 9^{k-n}\right) - p\left(9^{k-n}k_n\right)\right) + \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\cdot\left(p\left(9^{k-n}k_n + 9^{k-n}\right) - p\left(9^{k-n}k_n\right)\right)$$

$= M_1 + M_2$

Now we will get a lower bound for $M_1 + M_2$:

<u>First find a lower bound for M_1 $(k < n)$:</u>

Recall, given a linear function and 2 ordered pairs (x_1, y_1) and (x_2, y_2) which satisfy the function:

$$y_2 - y_1 = m(x_2 - x_1)$$

In a step-wise linear function, if $x_2 > x_1$, then $y_2 - y_1$ attains its least value when m is the smallest and $(x_2 - x_1)$ is the largest in that step.

Note that the smallest possible value of $p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)$ occurs when both $(9^{k-n}k_n + 9^{k-n})$ and $(9^{k-n}k_n)$ lie in the interval when $p(x) = 5 - 3x$, where the slope is at its least value (-3).

So the smallest value occurs when $\frac{p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)}{9^{k-n}} = -3$

So the smallest value occurs when $p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n) = -3(9^{k-n})$

So $M_1 = \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\left(p\left(9^{k-n}k_n + 9^{k-n}\right) - p\left(9^{k-n}k_n\right)\right)$ from above

$\geq \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}(-3)(9^{k-n})$ substituting in the smallest value for $p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)$

$= \frac{-3}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}(9^{k-n})$

$= \frac{-3}{2}\sum_{k=0}^{n-1}\left(\frac{9}{2}\right)^k\frac{1}{9^n}$

$= \frac{-3}{2\cdot 9^n}\sum_{k=0}^{n-1}\left(\frac{9}{2}\right)^k$ This is a finite geometric series, which we sum in the next step.

$= \frac{-3}{2\cdot 9^n}\left(\frac{1-\left(\frac{9}{2}\right)^n}{1-\left(\frac{9}{2}\right)}\right)$ Using algebra, we get the next step:

$= \frac{-3}{7\cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right)$ This is a Lower Bound for M_1

<u>Now we find a simplified expression for M_2 $(k \geq n)$:</u>

$M_2 = \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\cdot\left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right)$

Consider $(9^{k-n}k_n)$. This is the product of an odd number and an even number, and so it is even. So $p(9^{k-n}k_n)$=0.
Consider $(9^{k-n}k_n + 9^{k-n})$. This is the sum of an even number and an odd number, and so it is odd. So $p(9^{k-n}k_n + 9^{k-n})$= 1.

So $M_2 = \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}(1-0)$ Substituting in for $p(9^{k-n}k_n + 9^{k-n})$ and $p(9^{k-n}k_n)$ in above equation

$= \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k} = \frac{1}{2}\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n}$ Calculating the sum of an infinite series, r< 1

So $M_2 = \frac{1}{2^n}$

Now consider $\frac{\varphi_s(b_n) - \varphi_s(a_n)}{b_n - a_n} = \frac{M_1 + M_2}{9^{-n}}$

$= 9^n(M_1 + M_2)$

$\geq 9^n\left(\frac{-3}{7\cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right) + \frac{1}{2^n}\right)$ substituting in the LB for M_1 and M_2

$= \frac{3}{7} + \frac{4}{7}\left(\frac{9}{2}\right)^n$ This diverges to ∞ as $n \to \infty$.

So $\lim_{n\to\infty}\frac{\varphi_s(b_n) - \varphi_s(a_n)}{b_n - a_n}$ does not exist when $\widehat{k_n}$ is even and $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Therefore, $\varphi'_s(t)$ does not exist when $\widehat{k_n}$ is even, for $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Case ii: There are infinitely many odd $\widehat{k_n}$

Let $\{k_n\}$ be a subsequence of $\{\widehat{k_n}\}$ such that k_n is odd.

Let $\{a_n\}$ and $\{b_n\}$ be the corresponding sequences.

$\varphi_s(b_n)$ - $\varphi_s(a_n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^k b_n) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^k a_n)$

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\left(9^k(k_n 9^{-n} + 9^{-n})\right) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^k k_n 9^{-n})$ Sub in values for b_n & a_n

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^{k-n}k_n + 9^{k-n}) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(9^{k-n}k_n)$ using algebra

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot \left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right)$ combining into 1 series

$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right) + \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\cdot\left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right)$

$= M_1 + M_2$

Now we will get an upper bound for $M_1 + M_2$:

First find an upper bound for M_1 $(k < n)$:

Note that the largest possible value of $p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)$ occurs when both $9^{k-n}k_n + 9^{k-n}$ and $9^{k-n}k_n$ lie in the interval when $p(x) = 3x + 1$, where the slope is greatest (3). (See explanation provided above for lower bound.)

So the largest value occurs when $\frac{p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)}{9^{k-n}} = 3$

So the largest value occurs when $p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n) = 3(9^{k-n})$

From above, $M_1 = \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\cdot\left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right)$

$\leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\cdot 3(9^{k-n})$ Substituting in the UB for $p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)$

$= \frac{3}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\cdot (9^{k-n})$

$= \frac{3}{2}\sum_{k=0}^{n-1}\left(\frac{9}{2}\right)^k\frac{1}{9^n}$

$= \frac{3}{2\cdot 9^n}\sum_{k=0}^{n-1}\left(\frac{9}{2}\right)^k$ This is a finite geometric series, which we sum in the next step:

$= \frac{3}{2\cdot 9^n}\left[\frac{1-\left(\frac{9}{2}\right)^n}{1-\frac{9}{2}}\right]$ Use algebra to get the next step:

$= \frac{3}{7\cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right)$ This is an Upper Bound for M_1

Now we find a simplified expression for M_2 $(k \geq n)$:

Consider $(9^{k-n}k_n)$. This is the product of an odd number and an odd number, and so it is odd. So $p(9^{k-n}k_n)$=1.
Consider $(9^{k-n}k_n + 9^{k-n})$. This is the sum of an odd number and an odd number, and so it is even. So $p(9^{k-n}k_n + 9^{k-n})$= 0.

So M_2=$\frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k} \cdot \left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right)$

$=\frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}(0-1)$

$=\frac{1}{2}\sum_{k=n}^{\infty}\frac{-1}{2^k}$ This is an infinite geometric series, r<1 which we sum below

$=\frac{1}{2}\left[\frac{-\frac{1}{2^n}}{1-\frac{1}{2}}\right]$

$= \frac{-1}{2^n}$

So $M_2 = \frac{-1}{2^n}$

Now consider $\frac{\varphi_s(b_n)-\varphi_s(a_n)}{b_n - a_n}$ = $\frac{M_1+M_2}{9^{-n}}$

= $9^n(M_1 + M_2)$ Next we substitute in values for M_2 and an UB for M_1

$\leq 9^n\left(\frac{3}{7 \cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right) + \frac{-1}{2^n}\right)$ Use algebra to get the next step

= $\frac{-4}{7}\left(\frac{9}{2}\right)^n - \frac{3}{7}$ This diverges to $-\infty$ as $n \to \infty$.

So $\lim\limits_{n\to\infty}\frac{\varphi_s(b_n)-\varphi_s(a_n)}{b_n - a_n}$ does not exist when $\widehat{k_n}$ odd and $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Therefore, $\varphi'_s(t)$ does not exist when $\widehat{k_n}$ is odd and $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Therefore, $\varphi'_s(t)$ does not exist for $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Therefore, $\varphi_s(x)$ is not differentiable on (0,1), except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Since $\psi_s(x) = \varphi_s(3x)$, we can conclude that $\psi_s(x)$ is not differentiable on (0,1), except possibly when $x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m}$ for some m; $a_i = \{0, 1, 2, \ldots 8\}$.

III. Show that $\varphi_s(x)$ is not differentiable when $t = 0$:

For this proof, we construct a sequence $\{h_n\}$ and show that

$$\lim_{h_n \to 0} \frac{\varphi_s(h_n) - \varphi_s(0)}{h_n - 0} = \lim_{n \to \infty} \frac{\varphi_s(h_n) - \varphi_s(0)}{h_n - 0} \quad \text{does not exist}$$

$Note$: $\varphi(0) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(9^k \cdot 0) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k}(0) = 0$

Let $\{h_n\} = \left\{\frac{1}{9^n}\right\}$ (So $h_n \to 0$ as $n \to \infty$)

$\varphi_s(h_n) = \varphi_s\left(\frac{1}{9^n}\right) = \varphi_s(9^{-n})$

$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k}\, p(9^k \cdot 9^{-n})$

$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k}\, p(9^{k-n})$

$= \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{2^k}\, p(9^{k-n}) + \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\, p(9^{k-n})$ Note: if $k < n$, $9^{k-n} \leq \frac{1}{9}$, so $p(9^{k-n}) = 0$

$= \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\, p(9^{k-n})$ Note: 9^{k-n} is odd for $k \geq n$ as it is a positive integer power of 9, so $p(9^{k-n})$=1

$= \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}(1)$ This is the sum of an infinite geometric series, r<1

$= \frac{1}{2^n}$ Summing the series and multiplying by 1/2

So $\frac{\varphi_s(h_n) - \varphi_s(0)}{h_n - 0} = \frac{\frac{1}{2^n} - 0}{\frac{1}{9^n} - 0} = \left(\frac{9}{2}\right)^n$ which diverges to ∞ as $n \to \infty$.

Therefore, $\varphi'_s(0)$ does not exist. Similarly, $\psi'_s(0)$ does not exist.

IV. Show that $\varphi_s(x)$ and $\psi_s(x)$ are not differentiable when t=1:

For this proof, we construct a sequence $\{g_n\}$ and show that:

$$\lim_{g_n \to 1} \frac{\varphi_s(g_n)-\varphi_s(1)}{g_n-1} = \lim_{n\to\infty} \frac{\varphi_s(g_n)-\varphi_s(1)}{g_n-1} \quad \text{does not exist}$$

$\varphi_s(1)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^k\cdot 1)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^k)$ Note: $p(9^k)=1$ since 9^k is odd.

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}(1)=\frac{1}{2}(2)=1$ Summing up the infinite geometric series

Let $\{g_n\}=\left\{1-\frac{1}{9^n}\right\}$ (So $g_n \to 1$ as n→ ∞)

$\varphi_s(g_n) = \varphi_s(1-\frac{1}{9^n})$

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^k(1-\frac{1}{9^n}))$ Use algebra to get the next step

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^k-9^{k-n})$

$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}p(9^k-9^{k-n}) + \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}p(9^k-9^{k-n})$ express the above as 2 sums

Note: $p(9^k-9^{k-n})$ = 1 for $k<n$, since (9^k-9^{k-n}) is the difference of an odd number and a fraction $\leq \frac{1}{9}$
and $p(9^k-9^{k-n})$ = 0 for $k \geq n$, since (9^k-9^{k-n}) is the difference of 2 odd numbers, which is an even number

$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}(1) + \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}(0)$

$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}$ This is a finite geometric series

$= \frac{1}{2}\left(\frac{1-\left(\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}\right)$

$= 1-\frac{1}{2^n}$

So $\frac{\varphi_s(g_n)-\varphi_s(1)}{g_n-1} = \frac{\left(1-\frac{1}{2^n}\right)-1}{\left(1-\frac{1}{9^n}\right)-1} = \left(\frac{9}{2}\right)^n$ which diverges to ∞ as $n\to\infty$.

Therefore, $\varphi'_s(1)$ does not exist.

Similarly, $\psi'_s(1)$ does not exist.

Therefore, $\varphi'_s(x)$ and $\psi'_s(x)$ do not exist for $x \in [0,1]$, except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Therefore, Schoenberg's two functions, $\varphi_s(x)$ and $\psi_s(x)$, as defined above, are continuous and not differentiable on $[0,1]$, except possibly when

$$x = \frac{a_1}{9^1} + \frac{a_2}{9^2} + \cdots + \frac{a_m}{9^m} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots 8\}$$

Note: The above proof was adapted from the proofs found in the following documents:
!1). Ryder,J. (2011). *The Schoenberg Functions*. Word Press. Retrieved April 1, 2024, from https://caicedoteaching.wordpress.com/wp-content/uploads/2012/01/schoenberg_functions_ryder.pdf.
(2). Sagan,H.(1992). *Space-filling Curves*. New York. Springer-Verlag.
(3) Thim, J. (2003). *Continuous Nowhere Differentiable Functions* (2003:320 CIV). [Master's Thesis, Lulea University of Technology].
I have expanded the proof to provide more in-depth proofs of several of the steps.

Generalization of Proof

I have generalized the Schoenberg functions by making the following changes to the $\boldsymbol{p}$ function, and then proven that these generalized Schoenberg functions are continuous and not differentiable on [0,1] , with the possible exception as cited below.

- $p(3^{2k}x)$ *changed to* $p(j^{2k}x)$ $\quad j \in \{3, 5, 7, \ldots\}$
- $p(3^{2k+1}x)$ *changed to* $p(j^{2k+1}x)$

***Theorem: A generalization of Schoenberg's two functions, $\boldsymbol{\varphi_s(x)}$ and $\boldsymbol{\psi_s(x)}$, as defined below, are continuous and not differentiable on $\boldsymbol{[0,1]}$*, except possibly at**

$$\boldsymbol{x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots (j^2 - 1)\}}$$

$$\boldsymbol{\varphi_s(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k}x)} \qquad \boldsymbol{j \in \{3, 5, 7, \ldots\}}$$

$$\boldsymbol{\psi_s(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k+1}x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k} \cdot j \cdot x) = \varphi_s(jx)}$$

where

$$p(x) = 0 \qquad x \in \left[0, \frac{j-1}{2j}\right]$$

$p(x) = jx - \frac{j-1}{2} \qquad x \in \left[\frac{j-1}{2j},\ \frac{j+1}{2j}\right]$

$p(x) = 1 \qquad x \in \left[\frac{j+1}{2j},\ \frac{3j-1}{2j}\right]$

$p(x) = \frac{3j+1}{2} - jx \qquad x \in \left[\frac{3j-1}{2j},\ \frac{3j+1}{2j}\right]$

$p(x) = 0 \qquad x \in \left[\frac{3j+1}{2j},\ 2\right]$

and $p(x+2) = p(x)$ for all $x \in \mathcal{R}$. A graph of the function p follows.

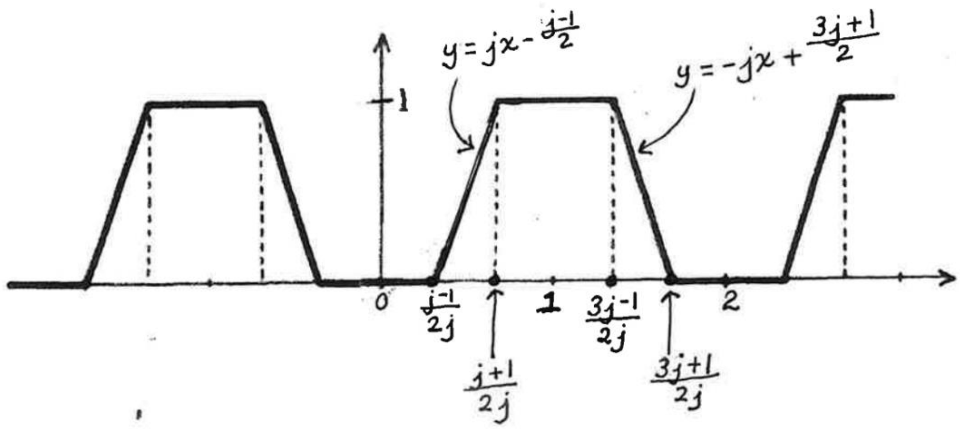


Figure 1: Generalized Schoenberg's p function

Note: If l is an odd integer, and $x \in \left(l - \frac{j-1}{2j},\ l + \frac{j-1}{2j}\right)$, then $p(x) = 1$

If l is an even integer, and $x \in \left(l - \frac{j-1}{2j},\ l + \frac{j-1}{2j}\right)$, then $p(x) = 0$

Proof:

I. Show that $\varphi_s(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k}x)$ is continuous on $[0,1]$

Let $\{s_k(x)\} = \left\{\frac{1}{2^k} p(j^{2k}x)\right\} \qquad k \in \mathcal{N}$

$Then\ \left|\frac{1}{2^k} p(j^{2k}x)\right| \leq \frac{1}{2^k}$ for each k (since $0 \leq p(j^{2k}x) \leq 1$)

So each term in the sequence is bounded above by $\frac{1}{2^k}$.

We know that $\sum_{k=0}^{\infty} \frac{1}{2^k}$ is an infinite geometric series with r< 1, and so it converges.

Since the series of upper bounds converges, then by the Weierstrass-M test,

$$\sum_{k=0}^{\infty} \frac{1}{2^k}\, p(j^{2k}x) \text{ converges uniformly on [0,1].}$$

Therefore, $\frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k}x)$ also converges uniformly on [0,1] to $\varphi_s(x)$.

Now define $\{f_n\} = \left\{\frac{1}{2}\sum_{i=1}^{n} S_i(x)\right\}_{n=1}^{\infty}$ to be the sequence of partial sums of $\varphi_s(x)$.

Then $\{f_n\}$ converges uniformly on [0,1] to $\varphi_s(x)$ since the corresponding series converges uniformly to $\varphi_s(x)$.

Note that each $S_k(x) = \frac{1}{2^k} p(j^{2k}x)$ is the product and composition of continuous functions, and therefore it is continuous.

So each $f_n(x)$is the sum of continuous functions, so it is continuous.

Since $\{f_n\}$ is a sequence of continuous functions which converges uniformly on [0,1] to $\varphi_s(x)$, we can conclude that:

$\varphi_s(x)$=$\frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k}x)$ is continuous on [0,1]. See IV in Background Information.

Similarly, $\psi_s(x)$ is also continuous on $[0,1]$.

II. Show that $\varphi_s(x)$ is nowhere differentiable on (0,1):

(Proof by contradiction)

Let $t\in (0,1)$. Assume that $\varphi'_s(t)$ exists.

Then by a previous lemma (*VI in Background Information*),

if $0 < a_n < t < b_n < 1,$ and $a_n \to t$ as $n \to \infty,$ and $b_n \to t$ as $n \to \infty,$ then

$$\lim_{n\to\infty} \frac{\varphi_s(b_n) - \varphi_s(a_n)}{b_n - a_n} = \varphi'_s(t)$$

We will construct 2 sequences, $\{a_n\}$ and $\{b_n\}$ which contradict our assumption:

Let $\widehat{k_n}$ = $[j^{2n}t]$ where $[x]$=the integer part of x; and t not of the form:

$$\frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m; \;\; a_i = \{0, 1, 2, \ldots (j^2 - 1)\};\; j \in \{3,5,7, \ldots\}$$

Note: This restriction on the form of t is required so that $[j^{2n}t]$ is always strictly less than $j^{2n}t$.

Let $\widehat{a_n} = \widehat{k_n} \cdot j^{-2n} = [j^{2n}t](j^{-2n})$

Let $\widehat{b_n} = \widehat{k_n} \cdot j^{-2n} + j^{-2n} = [j^{2n}t](j^{-2n}) + j^{-2n}$

Note: $\widehat{b_n} - \widehat{a_n} = j^{-2n}$

First we prove that $0 < \widehat{a_n} < t < \widehat{b_n} < 1$ and $\widehat{a_n} \to t$ as $n \to \infty$, and $\widehat{b_n} \to t$ as $n \to \infty$:

Proof of claim:

ia). Show that $\widehat{a_n} > 0$:

$\widehat{a_n} = [j^{2n}t](j^{-2n})$ by how we defined $\widehat{a_n}$

We know that $j^{-2n} > 0$ since $j \geq 3$

So we need to show that $[j^{2n}t] \geq 1$

So we need $j^{2n} \geq t^{-1}$

So we need $\ln j^{2n} \geq \ln t^{-1}$

So we need $2n \ln j \geq -1 \ln t$

So we need $n \geq \frac{-\ln t}{2\ln j}$

So whenever $n \geq \frac{-\ln t}{2\ln j}$, $[j^{2n}t] \geq 1$ *and* $\widehat{a_n} = [j^{2n}t](j^{-2n}) > 0.$

Since we are dealing with $n \to \infty$, we can restrict n in this manner.

ib). Show that $\widehat{a_n} < t$

$[j^{2n}t] < j^{2n}t$ Since the form of t is restricted.

So $\widehat{a_n} = [j^{2n}t](j^{-2n})$

$< (j^{2n}t)(j^{-2n})$

$= t$

So $\widehat{a_n} < t$

ic). Show that $a_n \to t$ as $n \to \infty$

We have shown that $\widehat{a_n} < t$

We know that $\lim_{n\to\infty} t = t$

Also, we know that $[j^{2n}t] > j^{2n}t - 1$

So $[j^{2n}t](j^{-2n}) > (j^{2n}t - 1)(j^{-2n})$

So $\widehat{a_n} > (j^{2n}t - 1)(j^{-2n})$

So $\widehat{a_n} > t - (j^{-2n})$

Now, $\lim_{n\to\infty} t - j^{-2n} = t$

Since $t - (j^{-2n}) < \widehat{a_n} < t$, we can conclude that $\lim_{n\to\infty} \widehat{a_n} = t$ (Squeeze Thm.)

iia) Show that $\widehat{b_n} < 1$:

We need $\widehat{b_n} = [j^{2n}t](j^{-2n}) + j^{-2n} < 1$

So we need $(j^{-2n})([j^{2n}t] + 1) < 1$

So we need $[j^{2n}t] + 1 < j^{2n}$

So we need $[j^{2n}t] < j^{2n} - 1$

Since $[j^{2n}t] < j^{2n}t$, we will first solve the following inequality:

$$j^{2n}t < j^{2n} - 1$$

$$j^{2n}t - j^{2n} < -1$$

$$j^{2n}(t - 1) < -1$$

$$j^{2n} > \frac{1}{1-t}$$

So $j^{2n} > (1-t)^{-1}$

$$\ln j^{2n} > \ln(1-t)^{-1}$$

$$2n \ln j > -1 \ln(1-t)$$

So $n > \frac{-1\ln(1-t)}{2\ln j}$

So whenever $n > \frac{-1\ln(1-t)}{2\ln j}$, we can say: $j^{2n}t < j^{2n} - 1$ and therefore $[j^{2n}t] < j^{2n} - 1$ and $\widehat{b_n} < 1$

So whenever $n > \frac{-1\ln(1-t)}{2\ln j}$, $\widehat{b_n} < 1$

Since we are dealing with $n \to \infty$, we can restrict n in this manner.

iib). Show that $\widehat{b_n} > t$:

$\widehat{b_n} = [j^{2n}t](j^{-2n}) + j^{-2n}$

We know that $[j^{2n}t] > j^{2n}t - 1$

So $[j^{2n}t](j^{-2n}) > (j^{2n}t - 1)(j^{-2n})$

So $[j^{2n}t](j^{-2n}) > t - (j^{-2n})$

So $[j^{2n}t](j^{-2n}) + (j^{-2n}) > t$

So $\widehat{b_n} > t$

iic). Show that $b_n \to t$ as $n \to \infty$:

Recall, $\widehat{b_n}$ = $[j^{2n}t](j^{-2n}) + j^{-2n}$

Since $[j^{2n}t] < j^{2n}t$, then

$[j^{2n}t](j^{-2n}) < (j^{2n}t)(j^{-2n})$

So $[j^{2n}t](j^{-2n}) + j^{-2n} < (j^{2n}t)(j^{-2n}) + j^{-2n} = t + j^{-2n}$

So $\widehat{b_n} < t + j^{-2n}$

Now, $\lim_{n\to\infty}(t + j^{-2n}) = t$

We have shown above that $\widehat{b_n} > t$, and we know that $\lim_{n\to\infty} t = t$

Since $t < \widehat{b_n} < t + j^{-2n}$, we can conclude that $\lim_{n\to\infty} \widehat{b_n} = t$. (Squeeze Thm.)

So we have shown that $0 < \widehat{a_n} < t < \widehat{b_n} < 1$ for large enough values of n and that $\lim_{n\to\infty} \widehat{a_n} = t$ and $\lim_{n\to\infty} \widehat{b_n} = t$.

Now we consider two groups of $\widehat{k_n}$: odd $\widehat{k_n}$ and even $\widehat{k_n}$

There are either infinitely many $\widehat{k_n}$ which are odd or infinitely many $\widehat{k_n}$ which are even

Case i: There are infinitely many even $\widehat{k_n}$

Let $\{k_n\}$ be a subsequence of $\{\widehat{k_n}\}$ such that k_n is even.

Let $\{a_n\}$ and $\{b_n\}$ be the corresponding sequences:

$a_n = k_n(j^{-2n})$ and $b_n = k_n(j^{-2n}) + j^{-2n}$

$\varphi_s(b_n) - \varphi_s(a_n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(j^{2k}b_n) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(j^{2k}a_n)$

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\left(j^{2k}(k_n j^{-2n} + j^{-2n})\right) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\left(j^{2k}k_n j^{-2n}\right)$ Sub in values of b_n & a_n

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\left(j^{2(k-n)}k_n\right)$

$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot \left(p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)\right)$ Combine series, factor out $\frac{1}{2^k}$

$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\left(p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)\right)$

$+ \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\cdot \left(p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)\right)$ express the series as 2 sums

$= M_1 + M_2$

Now we will get a lower bound for $M_1 + M_2$:

First find a lower bound for M_1 $(k < n)$:

Note that the smallest possible value of $p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)$ occurs when both $p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right)$ *and* $p\left(j^{2(k-n)}k_n\right)$ lie in the interval when $p(x) = \frac{3j+1}{2} - jx$, where the slope is at its smallest value $(-j)$. (*See proof in original proof for further explanation.)*

So the smallest value occurs when $\frac{p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)}{j^{2(k-n)}} = -j$

So the smallest value occurs when $p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right) = -j\left(j^{2(k-n)}\right)$

So $M_1 = \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\left(p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)\right)$ from above

$\geq \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}(-j)\left(j^{2(k-n)}\right)$ Substitute in the LB of $p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)$

$= \frac{-j}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\left(j^{2(k-n)}\right) = \frac{-j}{2}\sum_{k=0}^{n-1}\left(\frac{j^2}{2}\right)^k\frac{1}{j^{2n}}$

$= \frac{-j}{2\cdot j^{2n}}\sum_{k=0}^{n-1}\left(\frac{j^2}{2}\right)^k$ This is a finite geometric series which we sum below

$= \frac{-j}{2\cdot j^{2n}}\left(\frac{1-\left(\frac{j^2}{2}\right)^n}{1-\left(\frac{j^2}{2}\right)}\right)$

$= \left(\frac{-j}{(j^2-2)(j^{2n})}\right)\left(\left(\frac{j^2}{2}\right)^n - 1\right)$ this is a Lower Bound for M_1

Now we find a simplified expression for M_2 $(k \geq n)$:

$M_2 = \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\cdot \left(p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right)\right) - p\left(j^{2(k-n)}k_n\right)$

Consider $\left(j^{2(k-n)}k_n\right)$. This is the product of an odd number and an even number, and so it is even. So $p\left(j^{2(k-n)}k_n\right)$=0.

Consider $\left(j^{2(k-n)}k_n + j^{2(k-n)}\right)$. This is the sum of an even number and an odd number, and so it is odd. So $p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right)$=1

So $M_2 = \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\,(1-0)$ Substituting in values for $p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right)$ & $p\left(j^{2(k-n)}k_n\right)$

$= \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k} = \frac{1}{2}\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n}$

So $M_2 = \frac{1}{2^n}$

Now consider $\frac{\varphi_s(b_n)-\varphi_s(a_n)}{b_n - a_n} = \frac{M_1+M_2}{j^{-2n}}$

$= j^{2n}(M_1 + M_2)$

$\geq j^{2n}\left(\left(\frac{-j}{(j^2-2)(j^{2n})}\right)\left(\left(\frac{j^2}{2}\right)^n - 1\right) + \frac{1}{2^n}\right)$ Sub in value of M_2 & LB for M_1

$= \left(\frac{j^2}{2}\right)^n\left(\frac{j^2-j-2}{j^2-2}\right) + \frac{j}{j^2-2}$ Since $j \geq 3$, this diverges to ∞ as $n \to \infty$.

So $\lim_{n\to\infty}\frac{\varphi_s(b_n)-\varphi_s(a_n)}{b_n-a_n}$ does not exist when $\widehat{k_n}$ is even and $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m;\ \ a_i = \{0, 1, 2, \ldots (j^2-1)\}$$

Therefore, $\varphi'_s(x)$ does not exist when $\widehat{k_n}$ is even and $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m;\ \ a_i = \{0, 1, 2, \ldots (j^2-1)\}$$

Case ii: There are infinitely many odd $\widehat{k_n}$

Let $\{k_n\}$ be a subsequence of $\{\widehat{k_n}\}$ such that k_n is odd.

Let $\{a_n\}$ and $\{b_n\}$ be the corresponding sequences.

$\varphi_s(b_n)$ - $\varphi_s(a_n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(j^{2k}b_n) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(j^{2k}a_n)$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\Big(j^{2k}(k_n j^{-2n} + j^{-2n})\Big) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p(j^{2k}k_n j^{-2n})$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot p\big(j^{2(k-n)}k_n\big)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\cdot \Big(p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big) - p\big(j^{2(k-n)}k_n\big)\Big)$$

$$= \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{2^k}\cdot \Big(p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big) - p\big(j^{2(k-n)}k_n\big)\Big)$$

$$+ \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\cdot \Big(p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big) - p\big(j^{2(k-n)}k_n\big)\Big)$$

$$= M_1 + M_2$$

Now we will get an upper bound for $M_1 + M_2$:

First find an upper bound for M_1 $(k < n)$:

Note that the largest possible value of $p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big) - p\big(j^{2(k-n)}k_n\big)$ occurs when both $p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big)$ *and* $p\big(j^{2(k-n)}k_n\big)$ lie in the interval when $p(x) = jx - \frac{j-1}{2}$, where the slope is greatest (j)

So the largest value occurs when $\frac{p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)}{j^{2(k-n)}} = j$

So the largest value occurs when $p(j^{2(k-n)}k_n + j^{2(k-n)}) - p(j^{2(k-n)}k_n) = j\big(j^{2(k-n)}\big)$

From above, $M_1 = \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\Big(p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big) - p\big(j^{2(k-n)}k_n\big)\Big)$

$$\leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}(j)\big(j^{2(k-n)}\big)$$ Substituting in the largest value for $p\big(j^{2(k-n)}k_n + j^{2(k-n)}\big) - p\big(j^{2(k-n)}k_n\big)$

$$= \frac{j}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\big(j^{2(k-n)}\big)$$ Factoring out a j

$$= \frac{j}{2}\sum_{k=0}^{n-1}\left(\frac{j^2}{2}\right)^k\frac{1}{j^{2n}}$$ using algebra

$$= \frac{j}{2\cdot j^{2n}}\sum_{k=0}^{n-1}\left(\frac{j^2}{2}\right)^k$$ factoring out $\frac{1}{j^{2n}}$

$$= \frac{j}{2 \cdot j^{2n}} \left(\frac{1-\left(\frac{j^2}{2}\right)^n}{1-\left(\frac{j^2}{2}\right)} \right)$$ summing the finite geometric series

$$= \left(\frac{j}{(j^2-2)(j^{2n})}\right)\left(\left(\frac{j^2}{2}\right)^n - 1\right)$$ this is an Upper Bound for M_1

Now we find a simplified expression for M_2 $(k \geq n)$:

Consider $\left(j^{2(k-n)}k_n\right)$. This is the product of an odd number and an odd number, and so it is odd. So $p\left(j^{2(k-n)}k_n\right)$=1.

Consider $\left(j^{2(k-n)}k_n + j^{2(k-n)}\right)$. This is the sum of an odd number and an odd number, and so it is even. So $p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right)$=0

$$M_2 = \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k} \cdot \left(p\left(j^{2(k-n)}k_n + j^{2(k-n)}\right) - p\left(j^{2(k-n)}k_n\right)\right)$$ from above

$$= \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}(0-1) = \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}(-1) = -\frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k} = -\frac{1}{2}\left(\frac{1}{2^{n-1}}\right) = -\frac{1}{2^n}$$

So $M_2 = -\frac{1}{2^n}$

Now consider $\frac{\varphi_s(b_n)-\varphi_s(a_n)}{b_n - a_n} = \frac{M_1+M_2}{j^{-2n}}$

$$= j^{2n}(M_1 + M_2)$$

$$\leq j^{2n}\left(\left(\frac{j}{(j^2-2)(j^{2n})}\right)\left(\left(\frac{j^2}{2}\right)^n - 1\right) - \frac{1}{2^n}\right)$$ Subbing in value of M_2& UB of M_1

$$= -\left(\frac{j^2}{2}\right)^n\left(\frac{j^2-j-2}{j^2-2}\right) - \frac{j}{j^2-2}$$ This diverges to $-\infty$ as $n \to \infty$ since $j \geq 3$

So $\lim\limits_{n\to\infty} \frac{\varphi_s(b_n)-\varphi_s(a_n)}{b_n - a_n}$ does not exist when $\widehat{k_n}$ odd and $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \ldots (j^2-1)\}$$

Therefore, $\varphi'_s(t)$ does not exist when $\widehat{k_n}$ is odd and $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \dots (j^2 - 1)\}$$

Therefore, $\varphi'_s(t)$ does not exist for $x \in (0,1)$, except possibly when

$$x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \dots (j^2 - 1)$$

Therefore, $\varphi_s(t)$ is not differentiable on (0,1), except possibly when

$$x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \dots (j^2 - 1)\}$$

Since $\psi_s(t) = \varphi_s(jt)$, we can conclude that $\psi_s(t)$ is not differentiable on (0,1), except possibly when $x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}}$ for some m; $a_i = \{0, 1, 2, \dots (j^2 - 1)\}$

III. Show that $\varphi_s(x)$ is not differentiable when $x = 0$:

For this proof, we construct a sequence $\{h_n\}$ and show that:

$$\lim_{h_n \to 0} \frac{\varphi_s(h_n) - \varphi_s(0)}{h_n - 0} = \lim_{n \to \infty} \frac{\varphi_s(h_n) - \varphi_s(0)}{h_n - 0} \quad \text{does not exist}$$

Note: $\varphi_s(0) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k} \cdot 0) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(0) = 0$

Let $\{h_n\} = \left\{\frac{1}{j^{2n}}\right\}$ (So $h_n \to 0$ as $n \to \infty$)

$\varphi_s(h_n) = \varphi_s\left(\frac{1}{j^{2n}}\right)$

$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(j^{2k} \cdot j^{-2n})$

$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p\left(j^{2(k-n)}\right)$

$= \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{2^k} p\left(j^{2(k-n)}\right) + \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k} p\left(j^{2(k-n)}\right)$

Note: if $k < n$, $\left(j^{2(k-n)}\right) \leq \frac{1}{j^2} \leq \frac{1}{9} \leq \frac{j-1}{2j}$ *for* $j \geq 3$. *Thus* $p\left(j^{2(k-n)}\right) = 0$

$= \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k} p\left(j^{2(k-n)}\right)$ Note: $j^{2(k-n)}$ is odd for $k \geq n$, so $p\left(j^{2(k-n)}\right) = 1$

$= \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}(1) = \frac{1}{2^n}$

Now $\frac{\varphi_s(h_n)-\varphi_s(0)}{h_{n}-0} = \frac{\frac{1}{2^n}-0}{\frac{1}{j^{2n}}-0} = \left(\frac{j^2}{2}\right)^n$ Since $j \geq 3$, this diverges to ∞ as $n \to \infty$.

Therefore, $\varphi'_s(0)$ does not exist and similarly $\psi'_s(0)$ does not exist.

IV. Show that $\varphi_s(x)$ and $\psi_s(x)$ are not differentiable when t=1:

For this proof, we construct a sequence $\{g_n\}$ and show that:

$$\lim_{g_n \to 1} \frac{\varphi_s(g_n)-\varphi_s(1)}{g_n - 1} = \lim_{n\to\infty} \frac{\varphi_s(g_n)-\varphi_s(1)}{g_n - 1} \text{ does not exist}$$

Note: $\varphi_s(1) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(j^{2k}\cdot 1) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(j^{2k})$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}(1) \quad = \frac{1}{2}(2) = 1$$

Now we construct $\{g_n\}$ and examine $\varphi_s(g_n)$:

Let $\{g_n\} = \left\{1 - \frac{1}{j^{2n}}\right\}$ (So $g_n \to 1$ as n$\to \infty$)

$\varphi_s(g_n) = \varphi_s(1 - \frac{1}{j^{2n}})$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(j^{2k}(1 - \frac{1}{j^{2n}}))$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(j^{2k} - j^{2(k-n)})$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}p(j^{2k} - j^{2(k-n)}) + \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}p(j^{2k} - j^{2(k-n)})$$

Note: If $k < n$, $\left(j^{2(k-n)}\right) \leq \frac{1}{j^2} < \frac{j-1}{2j}$. So $p(j^{2k} - j^{2(k-n)})$ = 1 as $j^{2k} - j^{2(k-n)}$ is the difference of an odd number and a fraction less than $\frac{j-1}{2j}$.

Also if $k \geq n$, $p(j^{2k} - j^{2(k-n)})$ = 0, as $j^{2k} - j^{2(k-n)}$ is the difference of 2 odd integers

$$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}(1) + \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}(0)$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k} = 1 - \frac{1}{2^n}$$

So $\frac{\varphi_s(g_n)-\varphi_s(1)}{g_n-1} = \frac{\left(1-\frac{1}{2^n}\right)-1}{\left(1-\frac{1}{j^{2n}}\right)-1} = \left(\frac{j^2}{2}\right)^n$ which diverges to ∞ as $n \to \infty$, since $j \geq 3$

Therefore, $\varphi'_s(1)$ does not exist. Similarly, $\psi'_s(1)$ does not exist.

Therefore, $\varphi'_s(x)$ and $\psi'_s(x)$ do not exist for $x \in [0,1]$, except possibly when

$$x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}} \quad \text{for some } m; \quad a_i = \{0, 1, 2, \dots (j^2 - 1)\}$$

Therefore, Schoenberg's two functions, $\boldsymbol{\varphi_s(x)}$ and $\boldsymbol{\psi_s(x)}$, as generalized above, are continuous and not differentiable on $[\mathbf{0}, \mathbf{1}]$, except possibly when

$$\boldsymbol{x = \frac{a_1}{j^2} + \frac{a_2}{j^4} + \cdots + \frac{a_m}{j^{2m}}} \quad \textbf{for some } \boldsymbol{m}; \quad \boldsymbol{a_i = \{0, 1, 2, \dots (j^2 - 1)}$$

Chapter 4

Rudin's Function: Original Proof and Generalization

Walter Rudin was an Austrian-American mathematician. He published his proof of the existence of a real continuous function on the real line which is nowhere differentiable in 1953. His proof is given below:

Theorem: There exists a real continuous function on the real line which is nowhere differentiable.

Proof:

Part I: Construct the function

Define $\varphi(x) = |x|$ for $(-1 \le x \le 1)$ and extend the definition of $\varphi(x)$ to all real x by requiring that $\varphi(x+2) = \varphi(x)$

So φ is a periodic function, of period equal to 2, by virtue of its definition.

First we prove that $|\varphi(s) - \varphi(t)| \le |s - t|$ for all s and t:

Note: Either $|s-t| \ge 1$, or $|s-t| < 1$

Case 1: $|s-t| \ge 1$

We know that $0 \le \varphi(s) \le 1$ and $0 \le \varphi(t) \le 1$ by how we defined $\varphi(x)$.

Therefore, $|\varphi(s) - \varphi(t)| \le 1$.

Therefore, $|\varphi(s) - \varphi(t)| \le |s-t|$

Case 2: $|s-t| < 1$

Let s*= $\varphi(s)$ and t*= $\varphi(t)$. We consider 4 possibilities for Case 2:

i) $s = 2n + s^*$ ***and*** $t = 2n + t^*$

$$|s-t| = |(2n+s^*) - (2n+t^*)| = |s^* - t^*| = |\varphi(s) - \varphi(t)|$$

Therefore, $|\varphi(s) - \varphi(t)| = |s-t|$

ii) $\boldsymbol{s = 2n - s^*}$ ***and*** $\boldsymbol{t = 2n - t^*}$

$$|s-t|=|(2n-s^*)-(2n-t^*)| = |t^*-s^*| = |s^*-t^*| = |\varphi(s)-\varphi(t)|$$

Therefore, $|\varphi(s)-\varphi(t)| = |s-t|$

iii) $\boldsymbol{s = 2n + s^*}$ $\quad \boldsymbol{and} \quad$ $\boldsymbol{t = 2n - t^*}$

$$\begin{aligned} |s-t| &= |(2n+s^*)-(2n-t^*)| \\ &= |s^*+t^*| \\ &\geq |s^*-t^*| \qquad \textit{since } 0 \leq s^* \leq 1 \textit{ and } 0 \leq t^* \leq 1 \\ &= |\varphi(s)-\varphi(t)| \end{aligned}$$

Therefore, $|\varphi(s)-\varphi(t)| \leq |s-t|$

iv) $\boldsymbol{s = 2n + s^*}$ **and** $\mathbf{t} = \boldsymbol{2(n+1) - t^*}$

$$\begin{aligned} |s-t| &= |t-s| = |2(n+1)-t^* - (2n+s^*)| \\ &= |2-t^*-s^*| = |1-t^*+1-s^*| = |(1-t^*)+(1-s^*)| \end{aligned}$$

Now since $0 \leq t^* \leq 1$ *and* $0 \leq s^* \leq 1$, $(1-t^*) \geq 0$ *and* $(1-s^*) \geq 0$:

$$\begin{aligned} &\geq |(1-t^*)-(1-s^*)| \\ &= |s^*-t^*| = |\varphi(s)-\varphi(t)| \end{aligned}$$

Therefore, $|\varphi(s)-\varphi(t)| \leq |s-t|$

So we have shown that in all cases, $\boldsymbol{|\varphi(s)-\varphi(t)| \leq |s-t|}$

Next, we investigate whether or not φ *is continuous on the real numbers.*

Let $\varepsilon > 0$ *be given. Let* $t \in R$ *such that* $|t-x| < \varepsilon$.

Then by the above proof, $|\varphi(t)-\varphi(x)| \leq |t-x| < \varepsilon$.

Therefore, $\boldsymbol{\varphi(x)}$ ***is continuous on the real numbers by the definition of continuity.***

Next, we construct the following function, $f(x)$:

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \varphi(4^n x) = \sum_{k=1}^{\infty} \left(\tfrac{3}{4}\right)^{k-1} \varphi\left(4^{k-1} x\right)}$$

Part II: Show that $f(x)$ is continuous on $\mathscr{R}$

Expanding $f(x)$, we get:

$$f(x) = (\tfrac{3}{4})^0 \varphi(4^0 x) + (\tfrac{3}{4})^1 \varphi(4^1 x) + (\tfrac{3}{4})^2 \varphi(4^2 x) + \ldots.$$

The corresponding sequence is:

$$\{f_k(x)\} = \{\varphi(x), (\tfrac{3}{4})^1 \varphi(4^1 x), (\tfrac{3}{4})^2 \varphi(4^2 x), \ldots\}$$

Since $\varphi(x) \leq 1$, each term in the sequence is bounded (absolute value- wise) above by M_k:

$$|f_k(x)| = \left|\left(\tfrac{3}{4}\right)^{k-1} \varphi(4^{k-1} x)\right| = \left(\tfrac{3}{4}\right)^{k-1} |\varphi(4^{k-1} x)| \leq \left(\tfrac{3}{4}\right)^{k-1} = M_k$$

Now consider the series formed by the maximum values for each term: $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1}$. This is the sum of an infinite geometric series with r < 1, and therefore it converges.

By the Weierstrass-M Test, since $|f_k(x)| \leq M_k \quad (x \in R;\ k = 1, 2, 3, \ldots)$ and since $\sum_{k=1}^{\infty} M_k$ converges, we can conclude that

$$\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} (\tfrac{3}{4})^{k-1} \varphi(4^{k-1} x) \text{ converges uniformly to } f(x).$$

Now define $\{S_n\} = \{\sum_{i=1}^{n} f_i(x)\}_{n=1}^{\infty}$ to be the sequence of partial sums of $f(x)$.

Then $\{S_n\}$ converges uniformly to $f(x)$ since the corresponding series converges uniformly to $f(x)$.

Note that each $f_k(x) = \left(\frac{3}{4}\right)^{k-1} \varphi(4^{k-1} x)$ is the product and composition of continuous functions, and therefore it is continuous.

So each $S_n(x) = \{\sum_{i=1}^{n} f_i(x)\}_{n=1}^{\infty}$ is the sum of continuous functions, so it is continuous.

Since $\{S_n\}$ is a sequence of continuous functions which converges uniformly to $f(x)$, we can conclude that:

$f(x) = \sum_{k=1}^{\infty} (\frac{3}{4})^{k-1} \varphi(4^{k-1} x)$ is continuous on the real numbers. Therefore,

$f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^{n} \varphi(4^{n} x)$ is continuous on the real numbers.

Part III: Show that $f(x)$ is nowhere differentiable

First we construct a difference function, γ_n:

Fix a real number x and a positive integer m. Put $\boldsymbol{\delta_m} = \pm\frac{1}{2}(\mathbf{4^{-m}})$ *where the sign is chosen so that there is no integer between* $4^m x$ *and* $4^m(x + \delta_m)$. *This is possible, since* $4^m|\delta_m| = \frac{1}{2}$.

Define:

$$\boldsymbol{\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}}$$

We want to investigate the value of γ_n when $n > m$ and when $n \le m$.

*First **consider $n > m$:***

$4^n\delta_m = (4^{n-m})(4^m)(\delta_m) = (4^{n-m})\left(\pm\frac{1}{2}\right)$, *which is an even integer.*

So $\gamma_n = \dfrac{\varphi(4^n x + 4^n\delta_m) - \varphi(4^n x)}{\delta_m} = \dfrac{\varphi(4^n x + \text{even integer}) - \varphi(4^n x)}{\delta_m} = \dfrac{\varphi(4^n x) - \varphi(4^n x)}{\delta_m}$ *by our definition of φ*

$= 0$

So for $n > m$, $\gamma_n = 0$.

*Now consider **$n \le m$:***

$|\gamma_n| = \left|\dfrac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}\right| \le \left|\dfrac{(4^n(x+\delta_m)) - (4^n x)}{\delta_m}\right|$ *see part I of this proof*

$= \left|\dfrac{4^n\delta_m}{\delta_m}\right| = |4^n| = 4^n$

So $|\gamma_n| \le 4^n$ for $n \le m$.

Now we evaluate $|\gamma_m|$:

$|\gamma_m| = \left|\dfrac{\varphi(4^m(x+\delta_m)) - \varphi(4^m x)}{\delta_m}\right| = \left|\dfrac{\varphi(4^m x + 4^m\delta_m) - \varphi(4^m x)}{\delta_m}\right| = \left|\dfrac{\varphi(4^m\delta_m)}{\delta_m}\right|$ *Since there is no integer between* $(4^m x + 4^m\delta_m)$ *and* $\varphi(4^m x)$

$= \left|\dfrac{\varphi(\pm\frac{1}{2})}{(\pm\frac{1}{2})(4^{-m})}\right| = 4^m$

So $|\gamma_m| = 4^m$

Now we investigate the difference quotient $\left|\frac{\boldsymbol{f(x+\delta_m)-f(x)}}{\boldsymbol{\delta_m}}\right|$***:***

$$\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \left|\frac{\sum_{n=0}^{\infty} (\frac{3}{4})^n \varphi(4^n(x+\delta_m) - \sum_{n=0}^{\infty} (\frac{3}{4})^n \varphi(4^n x)}{\delta_m}\right|$$

$$= \left|\frac{\sum_{n=0}^{\infty} (\frac{3}{4})^n [\varphi(4^n(x+\delta_m) - \varphi(4^n x)]}{\delta_m}\right|$$

$$= \left|\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n (\gamma_n)\right|$$

$$= \left|\sum_{n=0}^{m} (\tfrac{3}{4})^n (\gamma_n) + \sum_{n=m+1}^{\infty} (\tfrac{3}{4})^n (\gamma_n)\right|$$ *write the above summation as 2 sums*

$$= \left|\sum_{n=0}^{m} (\tfrac{3}{4})^n (\gamma_n) + 0\right|$$ *since $\gamma_n = 0$ for $n > m$*

$$= \left|\sum_{n=0}^{m} (\tfrac{3}{4})^n (\gamma_n)\right|$$

$$= \left|(\tfrac{3}{4})^m (\gamma_m) + \sum_{n=0}^{m-1} (\tfrac{3}{4})^n (\gamma_n)\right|$$ *write the above summation as 2 sums*

$$\geq \left|(\tfrac{3}{4})^m (\gamma_m)\right| - \left|\sum_{n=0}^{m-1} (\tfrac{3}{4})^n (\gamma_n)\right|$$ *absolute value rules*

$$= (\tfrac{3}{4})^m |\gamma_m| - \left|\sum_{n=0}^{m-1} (\tfrac{3}{4})^n (\gamma_n)\right|$$

$$\geq (\tfrac{3}{4})^m |\gamma_n| - \sum_{n=0}^{m-1} \left|\left(\tfrac{3}{4}\right)^n\right| |\gamma_n|$$ *subtract a larger (or equal) quantity from the right*

$$\geq (\tfrac{3}{4})^m |\gamma_m| - \sum_{n=0}^{m-1} (\tfrac{3}{4})^n (4^n)$$ *subtract a larger (or equal) quantity from the right again since $|\gamma_n| \leq 4^n$ for $n \leq m$*

$$= \left(\frac{3^m}{4^m} \times 4^m\right) - \sum_{n=0}^{m-1} \left(\frac{3^n}{4^n} \times 4^n\right)$$ *Substitute 4^m for $|\gamma_m|$*

$$= 3^m - \sum_{n=0}^{m-1} 3^n$$ *This is a finite geometric series*

$$= 3^m - \left(\frac{1-3^m}{1-3}\right) = 3^m + \left(\frac{1-3^m}{2}\right) = \frac{1}{2}(3^m+1)$$

So $\left|\frac{\boldsymbol{f(x+\delta_m)-f(x)}}{\boldsymbol{\delta_m}}\right| \geq \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{3^m}+\mathbf{1})$

Consider $f'(x) = \lim\limits_{\delta_m \to 0} \frac{f(x+\delta_m)-f(x)}{\delta_m}$. *We have defined* $\delta_m = \left(\pm\frac{1}{2}\right)(4^{-m})$.

So as $m \to \infty, \delta_m \to 0$.

Also, we have found that $\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \geq \frac{1}{2}(3^m+1)$.

So as $m \to \infty, \frac{1}{2}(3^m+1) \to \infty$,

and therefore, $\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \to \infty$.

So as m→ ∞, $\delta_m \to 0$ *and* $\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \to \infty$.

Therefore, $f'(x) = \lim_{\delta_m \to 0} \frac{f(x+\delta_m)-f(x)}{\delta_m}$ *does not exist for all x.*

Therefore, $f(x)$ *is nowhere differentiable.*

Note: The above proof was adapted from the proof by Walter Rudin in Principles of Mathematical Analysis, Third Edition. New York, Mc-Graw-Hill, Inc., 1976. I have expanded the proof to include proofs of claims which were not proven in the original and more in depth proofs of several other steps.

Generalization of Proof:

I have generalized Rudin's function and proof by making the following changes:

- *Change the period to* $\mathbf{2a}$ *such that* $(-\boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{a})$; $\boldsymbol{a} \in \mathcal{N}$
- *Fix a number* $\boldsymbol{b}$ *such that* $\boldsymbol{b} = \mathbf{4}\boldsymbol{j}$ *where* $\boldsymbol{j}$ *is a natural number.*
- *Let* $\boldsymbol{c}$ *be a real number,* $1 < c < ba$
- *Construct a function* $f(x)$ *such that* $f(x) = \sum_{n=0}^{\infty} (\frac{c}{ba})^n \varphi((ba)^n x)$
- *Fix a natural number* $\boldsymbol{q}$ *such that* $\frac{\boldsymbol{b}}{\boldsymbol{q}} \in \{2, 4, 6, \ldots\}$, $q > \frac{1}{a}$, *and* $q \ge 2$, *and change* $\pm\frac{1}{2}$ *to* $\pm\frac{1}{q}$

*(*See notes at end of Part I regarding reasons for restrictions on variables.)*

Generalized Proof:

Part I: Construct the function

Let $\boldsymbol{a}$ *be a natural number.*

Fix a number $\boldsymbol{b}$ *such that* $\boldsymbol{b} = \mathbf{4}\boldsymbol{j}$ *where* $\boldsymbol{j}$ *is a natural number. (So* b *is a positive integer multiple of 4.)*

Let **c** *be a real number,* $\mathbf{1} < \boldsymbol{c} < \boldsymbol{ba}$

Define $\boldsymbol{\varphi}(\boldsymbol{x}) = |\boldsymbol{x}|$ ***for*** $(-\boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{a})$ ***and extend the definition of*** $\boldsymbol{\varphi}(\boldsymbol{x})$ ***to all real*** $\boldsymbol{x}$ ***by requiring that***

$$\boldsymbol{\varphi}(\boldsymbol{x} + \mathbf{2}\boldsymbol{a}) = \boldsymbol{\varphi}(\boldsymbol{x})$$

So φ *is a periodic function, of period equal to* $\mathbf{2}\boldsymbol{a}$, *by virtue of its definition.*

Note that $\varphi(0) = 0$, *so* $\varphi(0 + 2a) = 0$, *so* $\varphi(2a) = 0$, *and thus* φ*(even integer)=0*

Also, for all real numbers **s** *and* **t**,

$$|\boldsymbol{\varphi}(\boldsymbol{s}) - \boldsymbol{\varphi}(\boldsymbol{t})| \le |\boldsymbol{s} - \boldsymbol{t}|$$

(This fact can be proven similarly to the proof given for Rudin's original function.)

Next, we investigate whether or not φ is continuous on the real numbers.

Let $\varepsilon > 0$ be given. Let $t \in R$ such that $|t - x| < \varepsilon$.

Then by the above, $|\varphi(t) - \varphi(x)| \leq |t - x| < \varepsilon$.

Therefore, $\varphi(x)$ is continuous on the real numbers by the definition of continuity.

Next, we construct the following function, $f(x)$:

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \left(\frac{c}{ba}\right)^n \varphi((ba)^n x) = \sum_{k=1}^{\infty} \left(\frac{c}{ba}\right)^{k-1} \varphi\left((ba)^{k-1} x\right)}$$

*****<u>Reasons for restrictions on variables:</u>

We need $(ba)^n$ to be an integer multiple of $2a$. Therefore, b^n must be an integer multiple of 2, and a^n must be an integer multiple of a. So ***b*** must be an integer multiple of 2, and ***a*** must be a non-zero integer. I have chosen to restrict both ***a*** and ***b*** to the natural numbers.

We also need $\frac{b}{q}$ to be an integer multiple of *2a,* and we need $q \geq 2$, so we need $b \geq 4$

We need $c < ba$, in order for $\frac{c}{ba} < 1$, which we need in order to sum an infinite geometric series.

We need $c > 1$, in order for $\lim_{m\to\infty} \frac{1}{c-1}(c^m + 1)$ to be ∞. This is needed for the final part of the proof

Part II: Show that $f(x)$ is continuous on $\mathcal{R}$

Expanding $f(x)$, we get:

$f(x) = (\frac{c}{ba})^0\ \varphi((ba)^0 x) + (\frac{c}{ba})^1\ \varphi((ba)^1 x) + (\frac{c}{ba})^2\ \varphi((ba)^2 x) + \ldots.$

The corresponding sequence is:

$\{f_k(x)\} = \{\varphi(x),\ (\frac{c}{ba})^1 \varphi((ba)^1 x),\ (\frac{c}{ba})^2 \varphi((ba)^2 x), \ldots\}$

So the kth term in this sequence is: $f_k(x) = \left(\frac{c}{ba}\right)^{k-1} \varphi((ba)^{k-1} x)$

Since $\varphi(x) \leq a$, each term in the sequence is bounded (absolute value- wise) above by M_k:

$$|f_k(x)| = \left|\left(\frac{c}{ba}\right)^{k-1} \varphi((ba)^{k-1} x)\right| = \left(\frac{c}{ba}\right)^{k-1} |\varphi((ba)^{k-1} x)| \leq \left(\frac{c}{ba}\right)^{k-1} a = M_k$$

Now consider the series formed by the maximum values for each term: $\sum_{k=1}^{\infty} a\left(\frac{c}{ba}\right)^{k-1}$ *. This is the sum of an infinite geometric series with* r *< 1, and therefore it converges.*

By the Weierstrass-M Test, since $|f_k(x)| \leq M_k \quad (x \in R;\ k = 1, 2, 3, \ldots)$ *and since* $\sum_{k=1}^{\infty} M_k$ *converges, we can conclude that*

$$\sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \left(\frac{c}{ba}\right)^{k-1} \varphi((ba)^{k-1}x) \text{ converges uniformly to } f(x).$$

Now define $\{S_n\} = \{\sum_{i=1}^{n} f_i(x)\}_{n=1}^{\infty}$ to be the sequence of partial sums of $f(x)$.

Then $\{S_n\} = \{\sum_{i=1}^{n} f_i(x)\}_{n=1}^{\infty}$ converges uniformly to $f(x)$ since the corresponding series converges uniformly to $f(x)$.

Note that each $f_k(x) = \left(\frac{c}{ba}\right)^{k} \varphi((ba)^k x)$ is the product and composition of continuous functions, and therefore it is continuous.

So each $S_n(x)$is the sum of continuous functions, so it is continuous.

Since $\{S_n\}$ is a sequence of continuous functions which converges uniformly to $f(x)$, then by Theorem IV in Background Information, we can conclude that:

$f(x) = \sum_{n=0}^{\infty} \left(\frac{c}{ba}\right)^n \varphi((ba)^n x) = \sum_{k=1}^{\infty} \left(\frac{c}{ba}\right)^{k-1} \varphi((ba)^{k-1}x)$ is continuous on $\mathcal{R}$.

Part III: Show that $f(x)$ is nowhere differentiable

First we construct a difference function, $\boldsymbol{\gamma_n}$*:*

Fix a real number $\boldsymbol{x}$ *and a positive integer* $\boldsymbol{m}$.

Fix a natural number $\boldsymbol{q}$ *such that* $\frac{\boldsymbol{b}}{\boldsymbol{q}} \in \{2, 4, 6, \ldots\}$, $q > \frac{1}{a}$, *and* $q \geq 2$. *This would make* $\frac{1}{q} < a$ *and* $\frac{1}{q} \leq \frac{1}{2}$

Since $\frac{1}{q} < a$, *then* $\varphi\left(\frac{1}{q}\right) = \frac{1}{q}$ We need these restrictions on q for subsequent steps in the proof.

Put $\boldsymbol{\delta_m} = \pm\frac{\boldsymbol{1}}{\boldsymbol{q}}((\boldsymbol{ba})^{-\boldsymbol{m}})$ *where the sign is chosen so that there is no integer between* $(ba)^m x$ *and* $(ba)^m(x + \delta_m)$*. This is possible, since* $(ba)^m|\delta_m| = \frac{1}{q} \leq \frac{1}{2}$

Define:

$$\boldsymbol{\gamma_n = \frac{\varphi\big((ba)^n(x+\delta_m)\big) - \varphi((ba)^n x)}{\delta_m}}$$

We want to investigate the value of γ_n *when* $n > m$ *and when* $n \leq m$.

First ***consider*** $\boldsymbol{n > m}$:

$(ba)^n\delta_m = ((ba)^{n-m})\,((ba)^m)\,(\delta_m)$

$= ((ba)^{n-m})\left(\pm\frac{1}{q}\right)$

$= \pm\left[\left(\frac{b}{q}\right)(a)\right][(ba)^{n-m-1}]$ *=a multiple of* $2a$

(since $(ba)^n$ *is an integer multiple of 2a and* $\frac{b}{q}$ *is an integer multiple of 2a)*

So $\gamma_n = \dfrac{\varphi((ba)^n x + (ba)^n\delta_m) - \varphi((ba)^n x)}{\delta_m} = \dfrac{\varphi((ba)^n x + \mathbf{multiple\ of\ 2a}) - \varphi((ba)^n x)}{\delta_m}$

(substitute "$multiple\ of\ 2a$" for $(ba)^n\delta_m$ *)*

$= \dfrac{\varphi((ba)^n x) - \varphi((ba)^n x)}{\delta_m}$ *= 0* *By our definition of* φ: $\varphi(x + a\ multiple\ of\ 2a) = \varphi(x)$

So for $\boldsymbol{n > m}$***,*** $\boldsymbol{\gamma_n}$ ***= 0.***

Now consider $\boldsymbol{n \leq m}$:

$|\gamma_n| = \left|\dfrac{\varphi\big((ba)^n(x+\delta_m)\big) - \varphi((ba)^n x)}{\delta_m}\right| \leq \left|\dfrac{\big((ba)^n(x+\delta_m)\big) - ((ba)^n x)}{\delta_m}\right|$ *(See Part I of proof)*

$= \left|\dfrac{(ba)^n\delta_m}{\delta_m}\right|$

$= |(ba)^n| = (ba)^n$

So $\boldsymbol{|\gamma_n| \leq (ba)^n\ \ for\ n \leq m}$

Now we evaluate $|\boldsymbol{\gamma_m}|$:

$|\gamma_m| = \left|\frac{\varphi\left((ba)^m(x+\delta_m)\right) - \varphi((ba)^m x)}{\delta_m}\right| = \left|\frac{\varphi((ba)^m x+(ba)^m\delta_m) - \varphi((ba)^m x)}{\delta_m}\right|$

Now since there is no integer between $(ba)^m(x+\delta_m)$ and $(ba)^m x$, $\varphi((ba)^m x + (ba)^m\delta_m) = \varphi((ba)^m x) + \varphi((ba)^m\delta_m)$:

$= \left|\frac{\varphi(\,(ba)^m\delta_m)}{\delta_m}\right|$

$= \left|\frac{\varphi\left(\pm\frac{1}{q}\right)}{\left(\pm\frac{1}{q}\right)((ba)^{-m})}\right|$ *Substitute equal quantities in above*

$= \left|\frac{\frac{1}{q}}{\left(\pm\frac{1}{q}\right)((ba)^{-m})}\right|$ *Since* $\frac{1}{q} < a$, $\varphi\left(\pm\frac{1}{q}\right) = \frac{1}{q}$

$= (ba)^m$

So $|\boldsymbol{\gamma_m}| = (\boldsymbol{ba})^{\boldsymbol{m}}$

Now we investigate the difference quotient $\left|\frac{\boldsymbol{f(x+\delta_m) - f(x)}}{\boldsymbol{\delta_m}}\right|$:

$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\frac{\sum_{n=0}^{\infty}\left(\frac{c}{ba}\right)^n\varphi((ba)^n(x+\delta_m)) - \sum_{n=0}^{\infty}\left(\frac{c}{ba}\right)^n\varphi((ba)^n x)}{\delta_m}\right|$

$= \left|\frac{\sum_{n=0}^{\infty}\left(\frac{c}{ba}\right)^n[\varphi((ba)^n(x+\delta_m)) - \varphi((ba)^n x)]}{\delta_m}\right|$

$= \left|\sum_{n=0}^{\infty}\left(\frac{c}{ba}\right)^n(\gamma_n)\right|$

$= \left|\sum_{n=0}^{m}\left(\frac{c}{ba}\right)^n(\gamma_n) + \sum_{n=m+1}^{\infty}\left(\frac{c}{ba}\right)^n(\gamma_n)\right|$ *express above series as the sum of 2 series*

$= \left|\sum_{n=0}^{m}\left(\frac{c}{ba}\right)^n(\gamma_n) + 0\right|$ *since* $\gamma_n = 0$ *for* $n > m$

$= \left|\sum_{n=0}^{m}\left(\frac{c}{ba}\right)^n(\gamma_n)\right|$

$= \left|\left(\frac{c}{ba}\right)^m(\gamma_m) + \sum_{n=0}^{m-1}\left(\frac{c}{ba}\right)^n(\gamma_n)\right|$ *express above series as a sum*

$\geq \left|\left(\frac{c}{ba}\right)^m(\gamma_m)\right| - \left|\sum_{n=0}^{m-1}\left(\frac{c}{ba}\right)^n(\gamma_n)\right|$ *absolute value rules*

$= \left(\frac{c}{ba}\right)^m|\gamma_m| - \left|\sum_{n=0}^{m-1}\left(\frac{c}{ba}\right)^n(\gamma_n)\right|$ *c,b,a are positive*

$\geq \left(\frac{c}{ba}\right)^m|\gamma_m| - \sum_{n=0}^{m-1}\left|\left(\frac{c}{ba}\right)^n\right||\gamma_n|$ *subtract a larger (or equal) quantity from the right*

$\geq \ (\frac{c}{ba})^m|\gamma_m| - \sum_{n=0}^{m-1} \ (\frac{c}{ba})^n((ba)^n)$ *$n \leq m$, so $|\gamma_n| \leq (ba)^n$, so we are subtracting a larger quantity from the right again*

$= \left(\frac{c^m}{(ba)^m} \ \times (ba)^m\right) - \sum_{n=0}^{m-1} \left(\frac{c^n}{(ba)^n} \ \times (ba)^n\right)$ *substituting $(ba)^m$ for $|\gamma_m|$*

$= c^m - \sum_{n=0}^{m-1} c^n$ This is a finite geometric series

$= c^m - \left(\frac{1-c^m}{1-c}\right) = c^m + \left(\frac{1-c^m}{c-1}\right) = \frac{1}{c-1}(c^m + 1)$

So $\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \geq \frac{1}{c-1}(c^m + 1)$

Consider $f'(x) = \lim_{\delta_m \to 0} \frac{f(x+\delta_m)-f(x)}{\delta_m}$.

We have defined $\delta_m = \left(\pm\frac{1}{q}\right)((ba)^{-m})$. *So as* $m \to \infty, \delta_m \to 0$.

Also, we have found that $\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \geq \frac{1}{c-1}(c^m + 1)$.

So as $m \to \infty$, $\frac{1}{c-1}(c^m + 1) \to \infty$ since c> 1

and therefore, $\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \to \infty$.

So as $m \to \infty, \delta_m \to 0$ *and* $\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \to \infty$.

Therefore, $f'(x) = \lim_{\delta_m \to 0} \frac{f(x+\delta_m)-f(x)}{\delta_m}$ *does not exist for all x.*

Therefore, $f(x)$ is nowhere differentiable.

Chapter 5

Concluding Remarks

Continuous nowhere differentiable functions are more prevalent than one would think. Johan Thim [3] presents a topological proof that "Almost every function in C[0,1] (the set of continuous functions on [0,1]) is nowhere differentiable." In this paper, I have presented an in-depth study of three of these functions and attempted to generalize the parameters used. The three functions which I studied (Schwarz, Schoenberg, and Rudin) are all constructed from an infinite series. Schoenberg and Rudin also included periodic functions and fractals in their constructions, and Schoenberg constructed a space-filling curve. These characteristics – infinite series, periodic functions, fractals, and space-filling curves – are common in many constructions of continuous nowhere differentiable functions. Other functions have been constructed using techniques including purely geometric approaches (such as Koch's snowflake), infinite products (Wen), and topological approaches. (Thim [4])

The process of generalizing a function and proof requires an understanding of sequences series; an understanding of derivatives; and an understanding of the elements and mechanics of the proof and how changing a given number would affect the function and proof. Although I presented my successful attempts to generalize these functions and proofs, I often was unable to generalize one or more elements. Often the functions required integers as opposed to real numbers in their constructions; this can be seen in Rudin's function where the period must be defined as "$\boldsymbol{2a}$", $\boldsymbol{a}$ being an *integer*. I attempted unsuccessfully to generalize the period in Schoenberg's function; a simple change of the period from $\mathbf{2}$ to $\mathbf{4}$ required proofs of quantities being odd, even, even but not a multiple of 4, etc., all of which were cumbersome and seemingly impossible to carry out if using a variable instead of a constant.

Prior to the 19th century, continuous functions were thought to have derivatives at many, if not all, of their points. This view changed when mathematicians, beginning with Bernard Bolzano in 1830, began investigating and constructing functions which are nowhere differentiable on an interval. Bolzano based his function on a geometric construction. Since then, many other mathematicians have contributed to the body of functions which are continuous but nowhere differentiable, often employing techniques from Analysis. These functions today are relevant in the areas of fractals, chaos, and wavelets. (Thim [4]) Generalization of these functions aids in their application and use.

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